Uniform inference with general $AR(p)$ processes

Tassos Magdalinos
University of Southampton

Katerina Petrova
Universita' Ca' Foscari

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Introduction: Magdalinos and Petrova (2026)

- AR(1) process

$$x_t - \mu = \rho \left(x_{t-1} - \mu\right) + u_t$$

- Related regression models: Predictive Regression and Local Projection.
- Confidence intervals for ρ with uniform coverage over a parameter space:

$$\Theta = [-M, M], \quad \text{some } M > 0$$

- Critical regions for parameters in PR and LP models with correct asymptotic size uniformly over Θ
- Combined data-driven instrumentation procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - standard inference (testing and CIs) based on $\mathcal{N}(0, 1)$
  - uniform size
  - first distribution-free inference procedure in explosive case

This paper

- AR(p) process

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + u_t$$

- We wish to do inference on $\phi = (\phi_1, ..., \phi_p)'$

$$H_0 : H\phi_n = h, \quad \text{rank}(H) = q$$

- Wald statistic for testing H_0

$$W_n = \frac{1}{\hat{\sigma}_n^2} (H\tilde{\phi}_n - h)' H' \left[H \left(X' P_Z X \right)^{-1} H' \right]^{-1} (H\tilde{\phi}_n - h) .$$

- $\tilde{\phi}_n$ extension of the MP(2026) IV estimator
- Z extension of the MP(2026) instrument
- Target: get $W_n \to_d \chi^2(q)$

This paper

- AR(p) process
$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + u_t$$
- Roots of the characteristic polynomial are the eigenvalues of the companion matrix R:
$$sp(R) = \{\rho_1, \ldots, \rho_p\} \subset \mathbb{C}.$$
- Discontinuity in the asymptotics of the OLS estimator of $\phi = (\phi_1, \ldots, \phi_p)'$ according to the properties of $sp(R)$.
- AR(p) parameter space:
$$\Theta = \{\rho \in sp(R) : \text{there exist } M, \delta > 0 \text{ s.t. } |\rho| \leq M \text{ and } sp(R) \cap (-1+\delta, 1-\delta)^c \subset \mathbb{R}\}$$
- Uniform inference on ϕ requires limit distribution theory along
$$(\rho_{1n}, \ldots, \rho_{pn}) \to (\rho_1, \ldots, \rho_p). \tag{1}$$
- Rate of convergence and limit distribution theory depends on:

1. the AR category of each $\rho_{in} \to \rho_i$ in (1)
2. the multiplicity of each ρ_{in} in (1)

Categorisation of AR regions

- $sp(R) = \{\rho_1, \dots, \rho_p\}$

$$(\rho_{1n}, \dots, \rho_{pn}) \to (\rho_1, \dots, \rho_p).$$

- Let $c := \lim_{n\to\infty} n(|\rho_n| - 1)$ exist in $[-\infty, \infty]$.
- Then the eigenvalue ρ_n belongs to one of the following classes:

C(i) *near-stationary* if $c = -\infty$

C(ii) *LUR* if $c \in \mathbb{R}$

C(iii) *near-explosive* if $c = \infty$

Classes C(i)-C(iii) have *regular* and *oscillating* subclasses:
$C(i) = C_+(i) \cup C_-(i)$, etc

AR(p) companion matrix

- Companion matrix:

$$R = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_p \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Simple Jordan form:

$$\text{geometric multiplicity} = \text{algebraic multiplicity} \tag{2}$$

- (2) implies that R is diagonalisable *iff* it has distinct eigenvalues
- In this case, adaptation of MP(2026) is relatively straightforward: AR$(p) \simeq p$-AR(1) components
- (2) also implies that each group of repeated eigenvalues of R is contained in a single Jordan block which will have size equal to the number of times the eigenvalue is repeated
- Hence, knowing how many times each eigenvalue is repeated is equivalent to knowing the Jordan form of R (up to permutation)

Jordan form of companion matrix

- AR(2) in companion form:

$$\begin{aligned} \underline{x}_{n,t} &= R_n \underline{x}_{n,t-1} + \underline{u}_{n,t} \qquad R_n = T_n \Lambda_n T_n^{-1} \\ T_n^{-1} \underline{x}_{n,t} &= \Lambda_n T_n^{-1} \underline{x}_{n,t-1} + T_n^{-1} \underline{u}_{n,t} \end{aligned}$$

- Jordan matrix:

$$\Lambda_n = \begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \mathbf{1}\{\rho_{1n} \neq \rho_{2n}\} + \begin{bmatrix} \rho_n & 1 \\ 0 & \rho_n \end{bmatrix} \mathbf{1}\{\rho_{1n} = \rho_{2n} = \rho_n\}$$

- Matrix of (generalised) eigenvectors:

$$\begin{aligned} T_n^{-1} &= \frac{1}{\rho_{1n} - \rho_{2n}} \begin{bmatrix} 1 & -\rho_{2n} \\ -1 & \rho_{1n} \end{bmatrix} \mathbf{1}\{\rho_{1n} \neq \rho_{2n}\} \\ &\quad + \begin{bmatrix} \rho_n & -\rho_n^2 \\ 1 & 0 \end{bmatrix} \mathbf{1}\{\rho_{1n} = \rho_{2n} = \rho_n\} \end{aligned} \tag{3}$$

- When $\rho_{1n} \neq \rho_{2n}$, denote

$$S_n := (\rho_{1n} - \rho_{2n})\, T_n^{-1} = \begin{bmatrix} 1 & -\rho_{2n} \\ -1 & \rho_{1n} \end{bmatrix} \tag{4}$$

Asymptotic conditioning of eigenvalues

- Data driven criterion for "equality" of ρ_{1n} and ρ_{2n}
- How accurately can we achieve this?

$$\begin{aligned}
\hat{\rho}_{1n} - \hat{\rho}_{2n} &= \rho_{1n} - \rho_{2n} + \hat{\rho}_{1n} - \rho_{1n} - (\hat{\rho}_{2n} - \rho_{2n}) \\
&= \rho_{1n} - \rho_{2n} + O_p\left(n^{-1/2}\left(|\rho_{1n} - 1| \vee |\rho_{2n} - 1|\right)^{1/2}\right) \\
&= \rho_{1n} - \rho_{2n} + O_p\left(n^{-1/2}\left(|\hat{\rho}_{1n} - 1| \vee |\hat{\rho}_{2n} - 1|\right)^{1/2}\right) \\
&= \rho_{1n} - \rho_{2n} + O_p\left(l_n^{-1}\right)
\end{aligned}$$

- So $|\hat{\rho}_{1n} - \hat{\rho}_{2n}|$ can assess the magnitude of $|\rho_{1n} - \rho_{2n}|$ only up to a l_n^{-1} Pitman drift.
- If $|\rho_{1n} - \rho_{2n}| = O_p\left(l_n^{-1}\right)$, we design our criterion to give $\rho_{1n} = \rho_{2n}$ even if this may not hold exactly.

Asymptotic conditioning of eigenvalues

- Define the sequence

$$l_n := n^{1/2}\left(|\hat{\rho}_{1n}-1|^{-1/2} \wedge |\hat{\rho}_{2n}-1|^{-1/2}\right)$$

  and the events

$$\Delta_n = \{l_n |\hat{\rho}_{1n}-\hat{\rho}_{2n}| > L_n\}, \qquad \text{some } L_n \to \infty$$

- Since

$$l_n |\hat{\rho}_{1n}-\hat{\rho}_{2n}| = l_n |\rho_{1n}-\rho_{2n}| + O_p(1),$$

  we conclude that:

$$\mathbf{1}_{\Delta_n} \to_p 0 \;\text{ if } |\rho_{1n}-\rho_{2n}| = O\left(n^{-1/2}(|\rho_{1n}-1|^{1/2} \wedge |\rho_{2n}-1|^{1/2})\right) \quad (5)$$

- "Estimate" the Jordan form of R as

$$\begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \rho_n & 1 \\ 0 & \rho_n \end{bmatrix} \mathbf{1}_{\Delta_n^c}$$

- An error of considering $\rho_{1n} \neq \rho_{2n}$ when in fact $\rho_{1n} = \rho_{2n}$ occurs with prob. tending to 0
- We need to show that asymptotic theory under Δ_n^c works when $|\rho_{1n}-\rho_{2n}|$ satisfies (5)

AR(2) instrumentation

- AR(2) in companion form:

$$\begin{aligned} \underline{x}_{n,t} &= R_n \underline{x}_{n,t-1} + \underline{u}_{n,t} \qquad R_n = T_n \Lambda_n T_n^{-1} \qquad (6) \\ T_n^{-1} \underline{x}_{n,t} &= \Lambda_n T_n^{-1} \underline{x}_{n,t-1} + T_n^{-1} \underline{u}_{n,t} \end{aligned}$$

- Estimated Jordan form of R_n:

$$\hat{\Lambda}_n := \begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \rho_n & 1 \\ 0 & \rho_n \end{bmatrix} \mathbf{1}_{\Delta_n^c} \qquad (7)$$

- We are trying to build an instrument process that follows (6) as closely as possible subject to the MP(2026) instrumentation and information on the integration order provided by (7):

$$\underline{z}_{n,t} = \tilde{\Lambda}_n \underline{z}_{n,t-1} + \underline{\tilde{u}}_{n,t}$$

$$\tilde{\Lambda}_n := \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \varphi_n & 1 \\ 0 & \varphi_n \end{bmatrix} \mathbf{1}_{\Delta_n^c}$$

- φ_{1n}, φ_{2n} and φ_n are artificial AR roots chosen as in MP(2026).
- Selection of $\underline{\tilde{u}}_{n,t}$ requires estimation of T_n^{-1} by OLS in (3).

AR(2) instrumentation: root selection

- Instrument process:

$$\underline{z}_{n,t} = \tilde{\Lambda}_n \underline{z}_{n,t-1} + \underline{\tilde{u}}_{n,t}$$

$$\tilde{\Lambda}_n = \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \varphi_n & 1 \\ 0 & \varphi_n \end{bmatrix} \mathbf{1}_{\Delta_n^c}$$

- For $i \in \{1, 2\}$:

$$\varphi_{in} = \lambda_{in} \mathbf{1}_{F_{in}^+} + \lambda_{in}^- \mathbf{1}_{F_{in}^-} + \Lambda_{in} \mathbf{1}_{\bar{F}_{in}^+} + \Lambda_{in}^- \mathbf{1}_{\bar{F}_{in}^-}$$

where: $\lambda_{in} \in C_+ (i)$, $\lambda_{in}^- \in C_- (i)$, $\Lambda_{in} \in C_+ (iii)$, $\Lambda_{in}^- \in C_- (iii)$

- Events F_{in} defined as in MP(2026) for each eigenvalue ρ_{1n}, ρ_{2n}:

$$F_{in} = \{n (|\hat{\rho}_{in}| - 1) \leq 0\} \qquad \bar{F}_{in} = \{n (|\hat{\rho}_{in}| - 1) > 0\}$$

$$\begin{aligned} F_{in}^+ &= F_{in} \cap \{\hat{\rho}_{in} \geq 0\} & F_{in}^- &= F_{in} \cap \{\hat{\rho}_{in} < 0\} \\ \bar{F}_{in}^+ &= \bar{F}_{in} \cap \{\hat{\rho}_{in} \geq 0\} & \bar{F}_{in}^- &= \bar{F}_{in} \cap \{\hat{\rho}_{in} < 0\} \end{aligned}$$

AR(2) instrumentation: quasi-innovation selection

- Instrument process:

$$\underline{z}_{n,t} = \tilde{\Lambda}_n \underline{z}_{n,t-1} + \underline{\tilde{u}}_{n,t}$$

- Direction selection:

$$\begin{aligned} \hat{T}_n^{-1} &= \frac{1}{\hat{\rho}_{1n} - \hat{\rho}_{2n}} \begin{bmatrix} 1 & -\hat{\rho}_{2n} \\ -1 & \hat{\rho}_{1n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \hat{\rho}_{2n} & -\hat{\rho}_{2n}^2 \\ 1 & 0 \end{bmatrix} \mathbf{1}_{\Delta_n^c} \\ \hat{S}_n &= \left(\hat{\rho}_{1n} - \hat{\rho}_{2n}\right) \hat{T}_n^{-1} \end{aligned}$$

- $\underline{\tilde{u}}_{n,t} = \underline{\tilde{u}}_{n,t}^{(d)} \mathbf{1}_{\Delta_n} + \underline{\tilde{u}}_{n,t}^{(r)} \mathbf{1}_{\Delta_n^c}$
- Discrete eigenvalues:

$$\begin{aligned} \underline{\tilde{u}}_{n,t}^{(d)} &= \mathbb{I}_{F_n^+} \hat{S}_n \Delta \underline{x}_{n,t} + \mathbb{I}_{F_n^-} \hat{S}_n \nabla \underline{x}_{n,t} + \mathbb{I}_{\bar{F}_n^+ \cup \bar{F}_n^-} \hat{S}_n \underline{\hat{u}}_{n,t} \\ \mathbb{I}_{F_n^+} &= diag\left(\mathbf{1}_{F_{1n}^+}, \mathbf{1}_{F_{2n}^+}\right), \mathbb{I}_{\bar{F}_n^+ \cup \bar{F}_n^-} = diag\left(\mathbf{1}_{\bar{F}_{1n}^+ \cup \bar{F}_{1n}^-}, \mathbf{1}_{\bar{F}_{2n}^+ \cup \bar{F}_{2n}^-}\right) \end{aligned}$$

- Repeated eigenvalues:

$$\underline{\tilde{u}}_{n,t}^{(r)} = \hat{T}_n^{-1} \Delta \underline{x}_{n,t} \mathbf{1}_{F_{1n}^+} + \hat{T}_n^{-1} \nabla \underline{x}_{n,t} \mathbf{1}_{F_{1n}^-} + \hat{T}_n^{-1} \underline{\hat{u}}_{n,t} \mathbf{1}_{\bar{F}_{1n}^+ \cup \bar{F}_{1n}^-}$$

AR(2) with distinct characteristic roots

- $\underline{x}_t = R\underline{x}_{t-1} + \underline{u}_t$
- R diagonalisable: $X_t = S_n \underline{x}_t$

$$S_n = (\rho_{1n} - \rho_{2n})\, T_n^{-1} = \begin{bmatrix} 1 & -\rho_{2n} \\ -1 & \rho_{1n} \end{bmatrix} = \begin{bmatrix} S_{1n}' \\ S_{2n}' \end{bmatrix}$$

implies that

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \begin{bmatrix} X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ -u_t \end{bmatrix}$$

- MP(2022) instrumentation:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \end{bmatrix} + \begin{bmatrix} \tilde{u}_{1t} \\ -\tilde{u}_{2t} \end{bmatrix} \tag{8}$$

  - $n\left(|\hat{\rho}_{in}| - 1\right) \leq 0 \Longrightarrow \varphi_{in}$ near-stationary and $\tilde{u}_{it} = \Delta x_t$
  - $n\left(|\hat{\rho}_{in}| - 1\right) > 0 \Longrightarrow \varphi_{in}$ near-explosive and $\tilde{u}_{it} = \hat{u}_t$
- Feasibility: need to estimate the *directions* S_{1n} and S_{2n}

AR(2) with distinct characteristic roots

- Estimate the *directions*

$$X_{1t} \simeq \hat{S}_{1n}' \underline{x}_t \quad \text{and} \quad X_{2t} \simeq \hat{S}_{2n}' \underline{x}_t$$

where

$$\hat{S}_n = (\hat{\rho}_{1n} - \hat{\rho}_{2n})\, \hat{T}_n^{-1} = \begin{bmatrix} 1 & -\hat{\rho}_{2n} \\ -1 & \hat{\rho}_{1n} \end{bmatrix} = \begin{bmatrix} \hat{S}_{1n}' \\ \hat{S}_{2n}' \end{bmatrix}$$

- IV residuals:

$$\begin{aligned} \tilde{u}_{1t} &= \hat{S}_{1n}' \Delta \underline{x}_t \mathbf{1}_{F_{1n}^+} + \hat{S}_{1n}' \nabla \underline{x}_t \mathbf{1}_{F_{1n}^-} + \hat{u}_t \mathbf{1}_{\bar{F}_{1n}^+ \cup \bar{F}_{1n}^-} \\ \tilde{u}_{2t} &= \hat{S}_{2n}' \Delta \underline{x}_t \mathbf{1}_{F_{2n}^+} + \hat{S}_{2n}' \nabla \underline{x}_t \mathbf{1}_{F_{2n}^-} + \hat{u}_t \mathbf{1}_{\bar{F}_{2n}^+ \cup \bar{F}_{2n}^-} \end{aligned}$$

- Eigenvector estimation error $\hat{S}_{in}$ is controllable because it only depends on the estimation error of $\hat{\rho}_{in}$

Eigenvector estimation error

- IV residuals:

$$\begin{aligned}
\tilde{u}_{1t} &= \Delta X_{1t}\mathbf{1}_{F_{1n}^+} + \nabla X_{1t}\mathbf{1}_{F_{1n}^-} + \hat{u}_t\mathbf{1}_{\bar{F}_{1n}^+\cup\bar{F}_{1n}^-} - r_{1t}^{(n)} \\
\tilde{u}_{2t} &= \Delta X_{2t}\mathbf{1}_{F_{2n}^+} + \nabla X_{2t}\mathbf{1}_{F_{2n}^-} - \hat{u}_t\mathbf{1}_{\bar{F}_{2n}^+\cup\bar{F}_{2n}^-} + r_{2t}^{(n)}
\end{aligned}$$

where

$$r_{1t}^{(n)} = (\hat{\rho}_{2n} - \rho_{2n})\left(\Delta x_{t-1}\mathbf{1}_{F_{1n}^+} + \nabla x_{t-1}\mathbf{1}_{F_{1n}^-}\right) \quad (9)$$

$$r_{2t}^{(n)} = (\hat{\rho}_{1n} - \rho_{1n})\left(\Delta x_{t-1}\mathbf{1}_{F_{2n}^+} + \nabla x_{t-1}\mathbf{1}_{F_{2n}^-}\right) \quad (10)$$

- If ρ_{2n} is explosive and ρ_{1n} is stable, x_t is explosive, but:
    - the exponential decay of $\hat{\rho}_{2n} - \rho_{2n}$ controls (9)
    - the arbitrary decay of $\mathbf{1}_{F_{2n}^+}$ and $\mathbf{1}_{F_{2n}^-}$ controls (10) (Lemma 2 of MP(2026)).
- The above ensures the feasibility of the instrumentation in (8).

IV estimator (distinct characteristic roots)

- IV estimator $\tilde{\phi}_n$ of ϕ: $\tilde{\phi}_n = \left(\sum_{t=1}^n \underline{z}_{t-1}\underline{x}_{t-1}'\right)^{-1} \sum_{t=1}^n \underline{z}_{t-1} x_t$

$$T_n = (\rho_{1n} - \rho_{2n})^{-1} \begin{bmatrix} \rho_{1n} & \rho_{2n} \\ 1 & 1 \end{bmatrix}.$$

- Diagonal normalisation matrices $D_n(\rho_{1n}, \rho_{2n})$ and $D_n(\varphi_{1n}, \varphi_{2n})$ depending on the AR regime of ρ_{1n}, ρ_{2n}:

$$D_n T_n' \left(\tilde{\phi}_n - \phi\right) = \left(D_{nz}^{-1} \sum_{t=1}^n Z_{t-1} X_{t-1}' D_n^{-1}\right)^{-1} D_{nz}^{-1} \sum_{t=1}^n Z_{t-1} u_t$$

Theorem

When $\varphi_{1n} = r\varphi_{2n}$ for $r \in (0,1)$, the limit distribution of $\tilde{\phi}_n$ is given by

$$D_n T_n' \left(\tilde{\phi}_n - \phi\right) \to_d \mathcal{MN}\left(0, \sigma^2 \left(V_{xz}^{-1}\right)' V_{zz} V_{xz}^{-1}\right)$$

where $V_{zz} > 0$ and $\det(V_{xz}) \neq 0$ a.s.

AR(2) with repeated characteristic root

- ρ_n repeated characteristic root

$$J(\rho_n) = \begin{bmatrix} \rho_n & 0 \\ 1 & \rho_n \end{bmatrix}, \quad T^{-1}(\rho_n) = \begin{bmatrix} \rho_n & -\rho_n^2 \\ 1 & 0 \end{bmatrix}$$

and $X_t := T^{-1}(\rho_n)\underline{x}_t$, the Jordan decomposition gives

$$X_t = J(\rho_n) X_{t-1} + U_t, \quad U_t = u_t \begin{bmatrix} \rho_n \\ 1 \end{bmatrix} \tag{11}$$

- Instrument (11) by

$$Z_t = J(\varphi_n) Z_{t-1} + \tilde{u}_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix}$$

  - $F_n = \{n(|\hat{\rho}_n| - 1) \leq 0\} \Rightarrow \varphi_n$ near-stationary, $\tilde{u}_t = \Delta^2 x_t$
  - $\bar{F}_n = n(|\hat{\rho}_n| - 1) > 0 \Rightarrow \varphi_n$ near-explosive, $\tilde{u}_t = \hat{u}_t$

OLS limit theory with near-stationary repeated ch. root

- Fixed parameter AR(p): Chan and Wei (1988, AoS)
- Extension to $\rho_n \to 1$ in C(i):

$$D(\rho_n) = diag\left[\left(1-\rho_n^2\right)^{-1/2}, \left(1-\rho_n^2\right)^{-3/2}\right]$$

$$D(\rho_n)^{-1}\frac{1}{n}\sum_{t=1}^{n} X_{t-1}X_{t-1}' D(\rho_n)^{-1} \to_p \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} > 0 \qquad (12)$$

- Since $Z_t \simeq X_t$ when $(\rho_n - 1)/(\varphi_n - 1) \to \infty$ and Z_t behaves asymptotically as

$$Z_t = J(\varphi_n) Z_{t-1} + u_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} \text{ when } (\rho_n - 1)/(\varphi_n - 1) \to 0$$

(12) with $(D(\rho_n), X_{t-1})$ replaced by $(D(\varphi_n), Z_{t-1})$ establishes $Z'Z$ asymptotic instrument validity.

OLS limit theory with near-explosive repeated ch. root

- Fixed explosive parameter AR(p): Jeganathan (1988, AoS)
- Extension to $\rho_n \to 1$ in C(iii):

$$\left(\rho_n^2 - 1\right) D\left(\rho_n\right)^{-1} J\left(\rho_n\right)^{-n} \sum_{t=1}^{n} X_{t-1} X_{t-1}' J'\left(\rho_n\right)^{-n} D\left(\rho_n\right)^{-1} \to_d M\left(\xi_1, \xi_2\right) \tag{13}$$

$$\begin{aligned} M\left(\xi_1, \xi_2\right) &= \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_1^2 + \xi_2^2 \end{bmatrix} > 0 \\ \left(\xi_1, \xi_2\right)' &\sim \mathcal{N}\left(0, \sigma^2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right) \end{aligned}$$

- Since the instrument process Z_t behaves asymptotically as

$$Z_t = J\left(\varphi_n\right) Z_{t-1} + u_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix}$$

(13) with $\left(D\left(\rho_n\right), X_{t-1}\right)$ replaced by $\left(D\left(\varphi_n\right), Z_{t-1}\right)$ establishes $Z'Z$ asymptotic instrument validity.

Signal matrix summary

- Key result:

$$D_n^{-1}\frac{1}{n}\sum_{t=1}^{n}X_{t-1}Z_{t-1}'D\left(\varphi_n\right)^{-1}\rightarrow_d \Psi, \quad \det\left(\Psi\right)\neq 0 \; a.s.$$

where

$$D\left(\varphi_n\right)=diag\left[\left|1-\varphi_n^2\right|^{-1/2},\left|1-\varphi_n^2\right|^{-3/2}\right]$$

- D_n is a *non-diagonal stochastic* normalisation matrix satisfying

$$\left\|D_n\right\|\rightarrow\infty$$

- D_n depends stochastically only on the partial sum process $B_n\left(t\right)$ of u_t
- Guarantees asymptotic independence of

$$n^{-1/2}D\left(\varphi_n\right)^{-1}\sum_{t=1}^{n}Z_{t-1}u_t \quad \text{and} \quad \left[D_n, B_n\left(t\right)\right]$$

Wald statistic

- Under H_0, the Wald statistic satisfies

$$W_n = \nu_n' P_{\tilde{H}_n'} \nu_n, \quad \tilde{H}_n = H\left(T_n'\right)^{-1}\left(D_n'\right)^{-1}\left(V_{xz}^{(n)\prime}\right)^{-1}\left(V_{zz}^{(n)}\right)^{1/2}$$

- "Score":

$$\nu_n = \frac{1}{\hat{\sigma}_n}\left(V_{zz}^{(n)}\right)^{-1/2} n^{-1/2} D\left(\varphi_n\right)^{-1} \sum_{t=1}^{n} Z_{t-1} u_t \to_d N\left(0, I_p\right).$$

- Critical region: $\mathcal{R}_{n,\alpha} = \left\{W_n > \chi_q^2\left(1-\alpha\right)^{-1}\right\}$
- AR(p) parameter space:

$$\Theta = \{\rho \in sp\left(R\right) : \text{there exist } M, \delta > 0 \text{ s.t. } |\rho| \leq M \text{ and } sp\left(R\right) \cap \left(-1+\delta, 1-\delta\right)^c \subset \mathbb{R}\}$$

Theorem (?)

$\lim_{n\to\infty} \sup_{\theta\in\Theta} \mathbb{P}_\theta\left(\mathcal{R}_{n,\alpha}\right) = \alpha$

Conclusion

- Question marks:
  - approximate the IV limit distribution under Δ_n^c (non-trivial Jordan form) when eigenvalues are close but not equal
  - Done for local to unit roots $\rho_{1n} = 1 + c_1/n$ and $\rho_{2n} = 1 + c_2/n$
  - Scale up the analysis of

$$D_n^{-1} \frac{1}{n} \sum_{t=1}^{n} X_{t-1} Z'_{t-1} D(\varphi_n)^{-1} \rightarrow_d \Psi, \quad \det(\Psi) \neq 0 \; a.s.$$

    from the AR(2) to the AR(p) case
- The method provides inference for AR(p) processes without knowledge of:
  - type of eigenvalues (stable/unit/explosive)
  - integration order of the AR(p) ranging from I(0) to I(p)
- Extension to VAR(p) interesting but difficult: in general, we lose the advantage of not having to worry about convergence of sequences of eigenvectors...