Uniform inference with general autoregressive processes

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Introduction

- We consider a predictive regression setup:

$$y_t = \mu + \beta x_{t-1} + \varepsilon_t \tag{1}$$

with x_t generated by stochastic difference equation

$$x_t = \rho_n x_{t-1} + u_t \tag{2}$$

$v_t = (\varepsilon_t, u_t)'$ $\mathcal{F}_t$-martingale difference with $\mathbb{E}_{\mathcal{F}_{t-1}}(v_t v_t') = \Sigma_v > 0$ and $\left(\|v_t\|^2\right)_{t\geq 1}$ UI.

- The main issue: discontinuity in the rate of convergence and the asymptotic distribution of the OLS estimators for ρ_n and β in different autoregressive regions
- Practitioners run the risk of constructing invalid confidence sets for ρ_n and β

Empirical Relevance

- Important motivating example: series for inflation; large degree of memory and persistence over time
  $\Rightarrow$ Inference (hypothesis tests and CIs) in models with inflation on RHS (both structural and reduced-form) can be invalid if incorrect persistence assumptions imposed
- Other relevant applications:
  - Testing for predictability in financial returns (Fama and French (1988)) where explanatory variables included very persistent: E/P or D/P ratio
  - Testing for asset price bubbles which display explosive behaviour during their formation (e.g. Phillips, Shi and Yu (2015))
  - Constructing CIs for epidemiological parameters such as the BRN in SIR-type models, series for infected individuals displays exponential growth at outbreak

Contributions

- AIM OF THE PAPER - provide an inference procedure that is robust and delivers valid inference for both ρ_n and β with uniform size for the parameter space

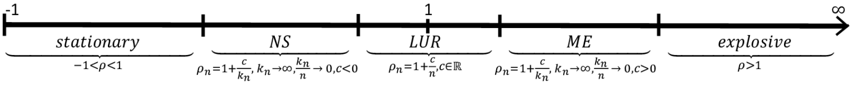
$$\rho_n \to \rho \in (-\infty, \infty)$$

- Contributions of the paper:
  - New endogenously constructed instruments - valid inference in explosive and unit root regions $(-\infty, -1] \cup [1, \infty)$
  - Combine new instruments with Phillips and Magdalinos (2009) to cover $(-\infty, \infty)$
- Combined data-driven procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - standard inference (testing and CIs) based on $\mathcal{N}(0, 1)$
  - uniform size
  - first distribution-free inference procedure in explosive case
- For presentational convenience: $\rho_n \to \rho \in (-1, \infty)$

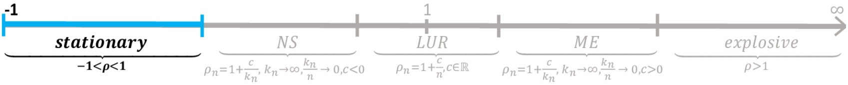
Plan for the Talk

- Motivation and different persistence regions
- Assumptions and Main Results
- Monte Carlo Simulations
- Applications to
  - SIR covid-19 model
  - bubbles in Magnificent 7 Tech stocks

Motivation

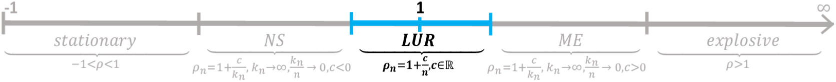

-1
1
∞
stationary
$-1<\rho<1$
NS
$\rho_n=1+\frac{c}{k_n}, k_n\to\infty, \frac{k_n}{n}\to 0, c<0$
LUR
$\rho_n=1+\frac{c}{n}, c\in\mathbb{R}$
ME
$\rho_n=1+\frac{c}{k_n}, k_n\to\infty, \frac{k_n}{n}\to 0, c>0$
explosive
$\rho>1$

Motivation



- **Stationarity region**: $\rho_n \to \rho \in (-1, 1)$
- Standard inference on $\hat{\rho}_n^{OLS}$ and $\hat{\beta}_n^{OLS}$
- $\sqrt{n}$-consistent asymptotically normal OLS estimators
- t-statistic is asymptotically $\mathcal{N}(0, 1)$

Motivation



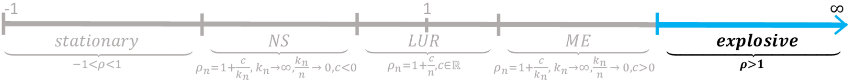
- **Local to unity region:** $\rho_n = 1 + c/n$, $c \in \mathbb{R}$
- FCLT to BMs: $n^{-1/2}\left[\sum_{j=1}^{\lfloor nt \rfloor} u_j, \sum_{j=1}^{\lfloor nt \rfloor} \varepsilon_j\right] \Rightarrow [B(t), B_\varepsilon(t)]$
- $n^{-1/2} x_{\lfloor nt \rfloor} \Rightarrow J_c(t) = \int_0^t e^{c(t-r)} dB(r)$ OU process w.r.t. B
- Phillips (1987 Biometrika), Chan and Wei (1987 AoS)

$$\left[n\left(\hat{\rho}_n^{OLS} - \rho_n\right), n\left(\hat{\beta}_n^{OLS} - \beta\right)\right] \to_d [K_c, L_c]$$

$$[K_c, L_c] = \left[\int_0^1 J_c(t)\, dB(t), \int_0^1 J_c(t)\, dB_\varepsilon(t)\right] / \int_0^1 J_c(t)^2\, dt$$

- n-consistent OLS, non-standard inference; no cv without knowing c
- Inconsistent OLS estimator for c: $\hat{c}_n^{OLS} - c = n\left(\hat{\rho}_n^{OLS} - \rho_n\right)$
- Campbell and Yogo (2006 JFE), Jansson and Moreira (2006 Ecta), Elliott, Müller and Watson (2015 Ecta)

Motivation



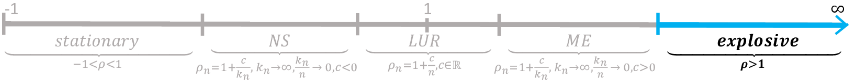
- **Explosive region**: $\rho \in (1, \infty)$ or $\rho_n \to \rho \in (1, \infty)$
- White (1958), Anderson (1959) AoMS: when $u_t \sim_{iid} (0, \sigma_u^2)$, and $x_0 = 0$

$$\frac{\rho^n}{(\rho^2 - 1)} \left(\hat{\rho}_n^{OLS} - \rho \right) \to_d \frac{Y_\infty}{Z_\infty}, \quad Z_\infty = \lim_{n\to\infty} \sum_{j=1}^{n} \rho^{-j} u_j \ a.s.$$

(Y_∞, Z_∞) independent, $Y_\infty =_d Z_\infty \sim \left(0, \sigma_u^2 / \left(\rho^2 - 1\right)\right)$

- No CLT: distribution of Y_∞ / Z_∞ driven by distribution of u_t
- Intuition: since Z_∞ is convergent *a.s.*, UAN fails
- First terms - non-negligible for Z_∞ distribution; last few terms carry entire $Y_\infty = \sum_{j=1}^{\infty} \rho^{-(n-j+1)} u_j$ distribution $\Longrightarrow$ t-statistic distribution

Motivation

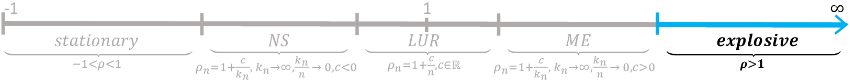


- If, in addition to $u_t \sim_{iid} \left(0, \sigma_u^2\right)$, $\left(u_t\right)_{t\in\mathbb{N}}$ is Gaussian then so are Y_∞, Z_∞ and Y_∞ / Z_∞ is a Cauchy distribution
- $\mathcal{N}\left(0,1\right)$ t-statistic only under i.i.d. Gaussian u_t with $x_0 = 0$
- In general, it is not normal and driven by the distribution of u_t
- Anderson (1959): example of non-convergent t-statistic
- Similar situation for inference on β:

$$\frac{\rho^n}{\rho^2-1}\frac{\sigma_u}{\sigma_\varepsilon}\left(\hat{\beta}_n^{OLS}-\beta\right)\to_d \frac{Y_\infty^\varepsilon}{Z_\infty}$$

  is $\mathcal{C}$ and t-statistic is $\mathcal{N}\left(0,1\right)$ only if $\left(\varepsilon_t, u_t\right) \sim i.i.d.\mathcal{N}$
- Since $Y_\infty^\varepsilon / Z_\infty$ is driven by the distribution of $\left(\varepsilon_t, u_t\right)$, no general semi-parametric inference is possible

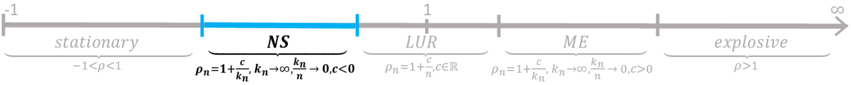
Motivation



- Illustration: consider the predictive regression setup, $\rho = 1.09$
- $[u_t, \varepsilon_t]' \sim i.i.d. \mathcal{N}(0, I_2)$, $t = 1, ..., n$
- Consider same sample, alter last observation $\varepsilon_n \sim \mathcal{N}\left(0, \sigma_\varepsilon^{*2}\right)$
- Table: empirical size of 95% two-sided t-test, based on 10,000 replications, under null $\beta = 0$:

	$\sigma_\varepsilon^* = 1$	$\sigma_\varepsilon^* = 2$	$\sigma_\varepsilon^* = 3$
$n = 20$	7.15%	10.14%	14.45%
$n = 50$	5.95%	10.29%	16.78%
$n = 100$	5.19%	10.65%	17.47%
$n = 500$	5.00%	10.89%	19.44%

Motivation



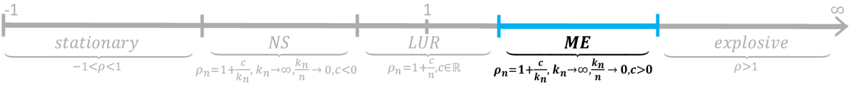
- **Near-stationary region**: $\rho_n = 1 + c/k_n, \ c < 0$
  $k_n \to \infty, \ k_n/n \to 0$
- Discontinuity in the OLS convergence rates: $\sqrt{n} \ll n \ll \rho^n$
- Phillips and Magdalinos (2007, JoE) bridge the gap:
- Sample moments satisfy a WLLN and a martingale CLT:

$$\frac{1}{nk_n} \sum_{t=1}^{n} x_{t-1}^2 \to_p \frac{\sigma_u^2}{-2c} \quad \text{and} \quad \frac{1}{\sqrt{nk_n}} \sum_{t=1}^{n} x_{t-1} u_t \to_d \mathcal{N}\left(0, \frac{\sigma_u^4}{-2c}\right)$$

- Ergodicity in autoregression obtains until LUR region

$$\sqrt{nk_n}\left[\hat{\rho}_n^{OLS} - \rho_n, \hat{\beta}_n^{OLS} - \beta\right] \to_d [1, \sigma_\varepsilon/\sigma_u]\, \mathcal{N}(0, -2c)$$

Motivation



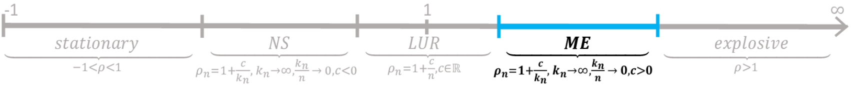
- **Mildly explosive region**: $\rho_n = 1 + c/k_n,\ c > 0$
  $k_n \to \infty,\ k_n/n \to 0$, Phillips and Magdalinos (2007, JoE)
- Similar to the explosive case

$$\frac{k_n \rho_n^n}{2c}\left(\hat{\rho}_n^{OLS} - \rho_n\right) = \frac{Y_n}{Z_n} + o_p(1)$$

  where $Z_n = k_n^{-1/2} \sum_{j=1}^n \rho_n^{-j} u_j,\ Y_n = k_n^{-1/2} \sum_{j=1}^n \rho_n^{-(n-j+1)} u_j$
- BUT, crucially, $\sum_{j=1}^n \rho_n^{-j} u_j$ and $\sum_{j=1}^n \rho_n^{-(n-j+1)} u_j$ diverge *a.s.*, so uniform asymptotic negligibility applies to the summand of (Z_n, Y_n) and a CLT applies:

$$(Z_n, Y_n) \to_d (Z, Y),\ Z, Y =_d \mathcal{N}\left(0, \sigma_u^2/2c\right),\ Z \perp Y.$$

Motivation

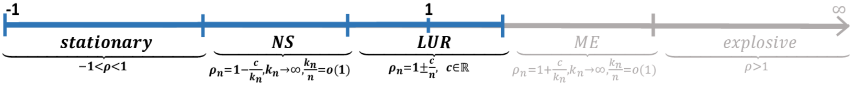


- The CLT on (Z_n, Y_n) yields

$$\frac{k_n\rho_n^n}{2c}\left(\hat{\rho}_n^{OLS}-\rho_n\right) = \frac{Y_n}{Z_n}+o_p(1)\to_d \mathcal{C}$$
$$\frac{k_n\rho_n^n}{2c}\frac{\sigma_u}{\sigma_\varepsilon}\left(\hat{\beta}_n^{OLS}-\beta\right) \to_d \mathcal{C}$$

- *Irrespective of the distribution of* (u_t):
  - Cauchy (mixed Gaussian) limit distribution for $\hat{\rho}_n^{OLS}$ and $\hat{\beta}_n^{OLS}$
  - Standard inference with asymptotically $\mathcal{N}(0,1)$ t-statistics

IVX instrumentation

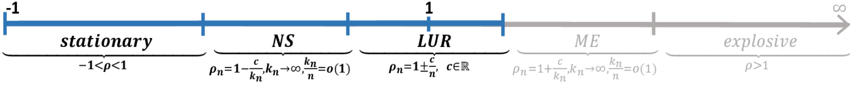


- Phillips and Magdalinos (2009): robust inference in the S-NS-LUR regions
- Employ Δx_t as quasi-residuals and construct an instrument

$$z_t = \rho_{nz} z_{t-1} + \Delta x_t \quad \rho_{nz} = 1 - 1/\kappa_n$$

  with *chosen* $\kappa_n \to \infty, \; \kappa_n/n \to 0$
- "IVX" estimators:
  - $\hat{\rho}_n^{IVX} = \frac{\sum_{t=1}^n z_{t-1} x_t}{\sum_{t=1}^n z_{t-1} x_{t-1}} = \rho_n + \frac{\sum_{t=1}^n z_{t-1} u_t}{\sum_{t=1}^n z_{t-1} x_{t-1}}$
  - $\hat{\beta}_n^{IVX} = \sum_{t=1}^n z_{t-1} \left(y_t - \bar{y}_n \right) / \sum_{t=1}^n z_{t-1} \left(x_{t-1} - \bar{x}_{n-1} \right)$

IVX instrumentation



- PM (2009) show that in S, NS, and LUR cases:

$$\sqrt{n\left(\kappa_n \wedge k_n\right)}\left(\hat{\rho}_n^{IVX} - \rho_n\right) \to_d \mathcal{MN}\left(0, \frac{\sigma_u^4}{2}\frac{1}{\Psi_c^2}\right)$$

$$\sqrt{n\left(\kappa_n \wedge k_n\right)}\left(\hat{\beta}_n^{IVX} - \beta_n\right) \to_d \mathcal{MN}\left(0, \frac{\sigma_u^2}{2}\frac{1}{\Psi_c^2}\sigma_\varepsilon^2\right)$$

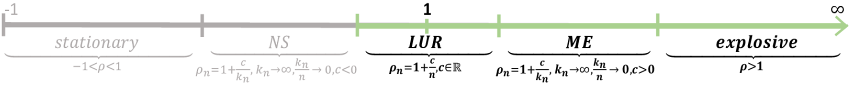
- The resulting t-statistics:

$$T_n\left(\hat{\rho}_n^{IVX}\right) = \sqrt{\sum_{t=1}^n x_{t-1}z_{t-1} / \hat{\sigma}_u^2 \sum_{t=1}^n z_{t-1}^2}\left(\hat{\rho}_n^{IVX} - \rho_n\right) \to_d \mathcal{N}(0,1)$$

$$T_n\left(\hat{\beta}_n^{IVX}\right) = \sqrt{\frac{\sum_{t=1}^n z_{t-1}\left(x_{t-1} - \bar{x}_{n-1}\right)}{\hat{\sigma}_\varepsilon^2 \sum_{t=1}^n \left(z_{t-1} - \bar{z}_{n-1}\right)^2}}\left(\hat{\beta}_n^{IVX} - \beta\right) \to_d \mathcal{N}(0,1)$$

in S, NS, and LUR cases

First contribution: LUR-ME-E regions

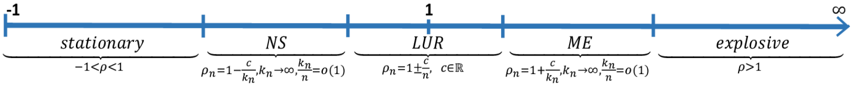


- Construct a new instrument in LUR, ME and E cases:

$$z_t = \rho_{nz} z_{t-1} + \hat{u}_t, \quad \rho_{nz} = 1 + 1/\kappa_n,$$

  $\hat{u}_t = \hat{u}_t^{OLS}$ and $\kappa_n \to \infty, \ \kappa_n/n \to 0$.
- Two important differences from IVX instrument of PM(2009):
  1. instrument behaves as a mildly explosive process
  2. instrument employs $\hat{u}_t^{OLS}$ instead of Δx_t
- We establish asymptotic $\mathcal{MN}$ of resulting IV estimators $\hat{\rho}_n^{IV}$ and $\hat{\beta}_n^{IV}$ and asymptotic $\mathcal{N}(0,1)$ of the t-statistic
- First *distribution-free* inference procedure in *explosive* region
- Procedure also valid in ME-LUR regions

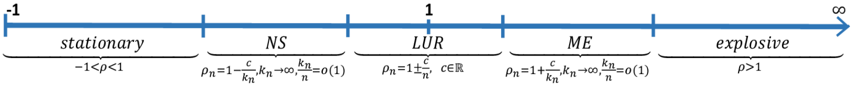
Second contribution: robust inference in all regions



- Combine the new instrument (LUR-ME-E regions) with the PM(2009) IVX instrument (S-NS-LUR regions) to have a testing procedure valid in all regions
- Key idea: since both instruments work in LUR case, only need information from data on whether x_t is S-NS or ME-E, i.e. when $k_n/n \to 0$ for $\rho_n = 1 + c/k_n$, we need info on $sign(c)$
- Such information available from $\hat{\rho}_n^{OLS}$:

$$n\left(\hat{\rho}_n^{OLS} - 1\right) \to_p sign(c) \times \infty$$

Second contribution: robust inference in all regions



- LUR region: $n\left(\hat{\rho}_n^{OLS}-1\right)=c+n\left(\hat{\rho}_n^{OLS}-\rho_n\right)\to_d c+K_c$
- Construct a combined instrument: $z_t=\rho_{nz}z_{t-1}+\tilde{u}_t$

$$\begin{aligned}\rho_{nz} &= 1-\frac{1}{\kappa_{1n}}\mathbf{1}\left\{n\left(\hat{\rho}_n^{OLS}-1\right)\leq 0\right\}+\frac{1}{\kappa_{2n}}\mathbf{1}\left\{n\left(\hat{\rho}_n^{OLS}-1\right)>0\right\}\\ \tilde{u}_t &= \Delta x_t\mathbf{1}\left\{F_n\right\}+\hat{u}_t\mathbf{1}\left\{\bar{F}_n\right\},\end{aligned}$$

- Since $\mathbf{1}\left\{F_n\right\}\to_p 1$ in S-NS region and $\mathbf{1}\left\{F_n\right\}\to_p 0$ in ME-E region, z_t behaves as a NS process in the S-NS regions and as a ME process in the ME-E regions
- In the LUR region, we show z_t: combination of the two instruments that **preserves the asymptotic mixed Gaussianity** of the IV estimators for ρ and β

Assumptions

$$\begin{aligned} y_t &= \mu + \beta x_{t-1} + \varepsilon_t \\ x_t &= \rho_n x_{t-1} + u_t \\ v_t &= (\varepsilon_t, u_t)' \end{aligned}$$

1. AR parameter space: $\rho \in (-1, \infty)$
   $(\rho_n)_{n \in \mathbb{N}}$ real sequence satisfying $\rho_n \to \rho \in (-1, \infty)$
2. Conditionally homoskedastic MD innovations:
   $$\mathbb{E}_{\mathcal{F}_{t-1}}(v_t) = 0 \;\; \mathbb{E}_{\mathcal{F}_{t-1}}(v_t v_t') = \Sigma_v > 0 \;\text{ for all } t$$
   and $\left(\|v_t\|^2\right)_{t \geq 1}$ is a UI sequence.
   Additionally, when $\rho_n \to \rho > 1$: $\liminf_{t\to\infty} \mathbb{E}_{\mathcal{F}_{t-1}} |u_t| > 0$
3. Initial condition:
   $x_0(n) = \max\left\{O_p(1), o_p\left(\min\left(|\rho_n - 1|^{-1/2}, n^{1/2}\right)\right)\right\}$

Categorisation of AR regions: Assumption 1*

1* x_t belongs to one of the following classes:

- C(i) *S-NS* processes if $n(\rho_n - 1) \to -\infty$
- C(ii) *LUR* processes if $n(\rho_n - 1) \to c \in \mathbb{R}$
- C(iii) *ME-E* processes if $n(\rho_n - 1) \to \infty$

Classes C(i)-C(iii) partition the parameter space when $\rho_n \to \rho \in (-1, \infty)$

- Assumption 1* is stronger than Assumption (1)

Combined instrument approach

- Data driven instrument selection: $F_n = \left\{ n \left(\hat{\rho}_n^{OLS} - 1 \right) \leq 0 \right\}$

$$
\begin{aligned}
\rho_{nz} &= \varphi_{1n} \mathbf{1}_{F_n} + \varphi_{2n} \mathbf{1}_{\bar{F}_n}, \quad \varphi_{1n} = 1 - \frac{1}{\kappa_{1n}}, \; \varphi_{2n} = 1 + \frac{1}{\kappa_{2n}}, \\
\kappa_{1n}, \kappa_{2n} &\to \infty, \; \kappa_{1n}/n \to 0, \; \kappa_{2n}/n \to 0
\end{aligned}
$$

- Combined instrument process:

$$
z_t = \rho_{nz} z_{t-1} + \tilde{u}_t \qquad \tilde{u}_t = \Delta x_t \mathbf{1}_{F_n} + \hat{u}_t \mathbf{1}_{\bar{F}_n} \qquad \left(\hat{u}_t \equiv \hat{u}_t^{OLS} \right).
$$

- Orthogonal decomposition: $z_t = z_{1t} \mathbf{1}_{F_n} + z_{2t} \mathbf{1}_{\bar{F}_n}$

$$
z_{1t} = \varphi_{1n} z_{1t-1} + \Delta x_t \qquad z_{2t} = \varphi_{2n} z_{2t-1} + \hat{u}_t
$$

$$
z_t \sim \begin{cases} z_{1t} \mathbf{1}_{F_n}, & \text{if } x_t \text{ is S-NS} \\ z_{2t} \mathbf{1}_{\bar{F}_n}, & \text{if } x_t \text{ is ME-E} \end{cases} \sim \begin{cases} z_{1t}, & \text{if } x_t \text{ is S-NS} \\ z_{2t}, & \text{if } x_t \text{ is ME-E} \end{cases}
$$

- In the LUR case *both* z_{1t} and z_{2t} contribute.

Asymptotic mixed Gaussianity: S-NS case C(i)

Theorem

Let x_t be a process in C(i). Under Assumptions 2-3,

$$\left(n\left(1-\rho_n^2\varphi_{1n}^2\right)^{-1}\right)^{1/2}\left(\tilde{\rho}_n-\rho_n\right)\to_d \mathcal{N}\left(0,1\right).$$

- Rate of convergence:

$$1-\rho_n^2\varphi_{1n}^2 \sim 2\left(1-\varphi_{1n}\right)+\left(1-\rho_n\right)\left(1+\rho_n\right)$$

  so $\left(1-\rho_n^2\varphi_{1n}^2\right)^{-1} \asymp \min\left\{k_n,\kappa_{1n}\right\}$
- When $k_n \ll \kappa_{1n}$, the above result reduces to the (near-) stationary OLS limit distribution

Asymptotic mixed Gaussianity: ME-E case C(iii)

Theorem

Let (X, Y) *independent,* $Y =_d \mathcal{N}\left(0, \sigma_u^2\right)$. *Under Assumptions 2-3*

$$\frac{\left(\varphi_{2n}^2 - 1\right)^{1/2}}{\left(\rho_n \varphi_{2n} - 1\right)\left(\rho_n^2 - 1\right)^{1/2}} \rho_n^n \left(\tilde{\rho}_n - \rho_n\right) \to_d \frac{Y}{X} =_d \mathcal{MN}\left(0, \sigma_u^2 / X^2\right)$$

(i) $X =_d \mathcal{N}\left(0, \sigma_u^2\right)$ *if* $\rho_n \to 1$;
(ii) $X = \left(\rho^2 - 1\right)^{1/2}\left(\sum_{j=1}^{\infty} \rho^{-j} u_j + x_0\right) \neq 0$ *a.s. if* $\rho_n \to \rho > 1$.

- In the mildly explosive case Y/X is Cauchy distributed, rate $\rho_n^n k_n \sqrt{\min\left(k_n, \kappa_{2n}\right) / \max\left(\left(k_n, \kappa_{2n}\right)\right)}$
- Asymptotic mixed Gaussianity applies to the explosive case (ii) *without distributional assumption* on (u_t)!
- OLS exponential rate preserved, $\rho_n^n / \sqrt{\kappa_{2n}}$

Asymptotic mixed Gaussianity: LUR case C(ii)

Theorem

Under Assumptions 2-3, if $n\left(\rho_n - 1\right) \to c \in \mathbb{R}$,

$$\left(\sqrt{\kappa_{1n}}\mathbf{1}_{F_n} + \sqrt{\kappa_{2n}}\mathbf{1}_{\bar{F}_n}\right)\sqrt{n}\left(\tilde{\rho}_n - \rho_n\right) \to_d \mathcal{MN}\left(0, \Sigma_c^2\right)$$

where:

$$\Sigma_c^2 = \frac{2\sigma_u^4}{\left(\sigma_u^2 + J_c\left(1\right)^2\right)^2}\mathbf{1}_{F_c} + \frac{\sigma_u^2}{2}\frac{1}{J_c\left(1\right)^2}\mathbf{1}_{\bar{F}_c}$$

$$F_c = \mathbf{1}\left\{K_c + c < 0\right\}, \quad K_c = \int_0^1 J_c\left(r\right)dB\left(r\right) / \int_0^1 J_c^2\left(r\right)dr$$

- Both instrument components z_{1t} and z_{2t} contribute.
- Random normalisation: $F_n = \left\{n\left(\hat{\rho}_n^{OLS} - 1\right) \leq 0\right\}$

Predictive regression setup

- Similar asymptotic mixed normality results for the IV estimator $\tilde{\beta}_n$ for Cases C(i)-C(iii)
- In the predictive regression setup, Assumption 2 can be relaxed to allow u_t to be a stationary linear process
- We can allow for stationary conditional heteroskedasticity (e.g. Garch) in u_t and ε_t (HC st. errors for t-test)
- In ME-E regions, ε_t can also be serially correlated (provided a consistent estimator for the LR variance is used for the t-test)

Nonstationary oscillations

- Consider $\rho_n \to \rho \in (-\infty, \infty)$, transform *oscillating* process into *regular* $x_t^+ = (-1)^t x_t$ with $x_t^+ = |\rho_n| x_{t-1}^+ + (-1)^t u_t$
- The new combined instrument is $\tilde{z}_t = \rho_{nz} \tilde{z}_{t-1} + \tilde{u}_t$ with

$$\rho_{nz} = \varphi_{1n} \mathbf{1}_{F_n^+} + \varphi_{2n} \mathbf{1}_{\bar{F}_n^+} + \varphi_{1n}^- \mathbf{1}_{F_n^-} + \varphi_{2n}^- \mathbf{1}_{\bar{F}_n^-}$$

  where $\varphi_{1n}^-, \varphi_{2n}^- \to -1$ are oscillating roots and $n\left(\left|\varphi_{1n}^-\right| - 1\right) \to -\infty$, $n\left(\left|\varphi_{2n}^-\right| - 1\right) \to \infty$
- Let $F_n = \{n(|\hat{\rho}_n| - 1) \leq 0\}$ and define:
  $F_n^+ = F_n \cap \{\hat{\rho}_n \geq 0\}$, $F_n^- = F_n \cap \{\hat{\rho}_n < 0\}$,
  $\bar{F}_n^+ = \bar{F}_n \cap \{\hat{\rho}_n \geq 0\}$ and $\bar{F}_n^- = \bar{F}_n \cap \{\hat{\rho}_n < 0\}$
- $\tilde{u}_t = \Delta x_t \mathbf{1}_{F_n^+} + \nabla x_t \mathbf{1}_{F_n^-} + \hat{u}_t \mathbf{1}_{\bar{F}_n^+} + \hat{u}_t \mathbf{1}_{\bar{F}_n^-}$ with $\nabla x_t := x_t + x_{t-1}$
- We establish AMG property of $\tilde{\rho}_n$ and $\tilde{\beta}$, and $\mathcal{N}(0,1)$ t-statistic (with uniform size)

Uniform inference under Assumption 1

- t-statistic based on $\tilde{\rho}_n$:

$$T_n\left(\tilde{\rho}_n\right) = \sqrt{\sum_{t=1}^n x_{t-1} z_{t-1} / \hat{\sigma}_u^2 \sum_{t=1}^n z_{t-1}^2}\left(\tilde{\rho}_n - \rho_n\right) \qquad (3)$$

- Confidence interval for ρ_n: $I_n\left(\tilde{\rho}_n, \alpha\right) = \left[\tilde{\rho}_n \pm c_n\left(\alpha\right)\right]$

$$c_n\left(\alpha\right) = \Phi^{-1}\left(1 - \tfrac{\alpha}{2}\right) \hat{\sigma}_u / \sqrt{\sum_{t=1}^n x_{t-1} z_{t-1} / \sum_{t=1}^n z_{t-1}^2}$$

- Critical region: $\mathcal{R}_n = \left\{\left|T_n\left(\tilde{\rho}_n\right)\right| > \Phi^{-1}\left(1 - \alpha/2\right)\right\}$

Theorem

(i) The t-statistic in (3) satisfies $T_n\left(\tilde{\rho}_n\right) \to_d \mathcal{N}\left(0, 1\right)$
(ii) $\mathcal{R}_n$ *is asymptotically similar with correct asymptotic size:*
$\liminf_{n\to\infty} \inf_{\rho\in[-M,M]} \mathbb{P}_\rho\left(\mathcal{R}_n\right) = \limsup_{n\to\infty} \sup_{\rho\in[-M,M]} \mathbb{P}_\rho\left(\mathcal{R}_n\right) = \alpha.$
(iii) The CI satisfies $\lim_{n\to\infty} \inf_{\rho\in[-M,M]} \mathbb{P}_\rho\left[\rho \in I_n\left(\tilde{\rho}_n, \alpha\right)\right] = 1 - \alpha.$

Uniform inference under Assumption 1

- IVX estimator based on combined instrument:

$$\tilde{\beta}_n = \sum_{t=1}^{n} z_{t-1} \left(y_t - \bar{y}_n\right) / \sum_{t=1}^{n} z_{t-1} \left(x_{t-1} - \bar{x}_{n-1}\right)$$

- t-statistic based on $\tilde{\beta}_n$ $T_n\left(\tilde{\beta}_n\right) = \left(\hat{\sigma}_{\varepsilon}^2 \sum_{t=1}^{n} \left(z_{t-1} - \bar{z}_{n-1}\right)^2\right)^{-1/2} \left(\sum_{t=1}^{n} z_{t-1} \left(x_{t-1} - \bar{x}_{n-1}\right)\right)^{1/2} \left(\tilde{\beta}_n - \beta\right)$
- Confidence interval for β: $I_n\left(\tilde{\beta}_n, \alpha\right) = \left[\tilde{\beta}_n \pm c_n(\alpha)\right]$, $c_n(\alpha) = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \hat{\sigma}_{\varepsilon} / \left(\sum_{t=1}^{n} \left(z_{t-1} - \bar{z}_{n-1}\right)^2\right)^{-1/2} \left(\sum_{t=1}^{n} z_{t-1} \left(x_{t-1} - \bar{x}_{n-1}\right)\right)^{1/2}$
- CR: $\tilde{\mathcal{R}}_n = \left\{\left|T_n\left(\tilde{\beta}_n\right)\right| > \Phi^{-1}\left(1 - \alpha/2\right)\right\}$

Theorem

(i) The t-statistic satisfies $T_n\left(\tilde{\beta}_n\right) \to_d \mathcal{N}(0,1)$
(ii) $\tilde{\mathcal{R}}_n$ *is asymptotically similar with correct asymptotic size:*
$\liminf_{n\to\infty} \inf_{\rho\in[-M,M]} \mathbb{P}_\rho\left(\tilde{\mathcal{R}}_n\right) = \limsup_{n\to\infty} \sup_{\rho\in[-M,M]} \mathbb{P}_\rho\left(\tilde{\mathcal{R}}_n\right) = \alpha.$
(iii) The CI satisfies $\lim_{n\to\infty} \inf_{\rho\in[-M,M]} \mathbb{P}_\rho\left[\beta \in I_n\left(\tilde{\beta}_n, \alpha\right)\right] = 1 - \alpha.$

Uniform asymptotic size of the t-statistic

- Test size uniform over the AR parameter space
- Subsequential argument to go from Assumption 1* to Assumption 1: $\rho_n \to \rho \in (-1, \infty)$
- For every subsequence, there exists a further subsequence (ρ_{m_n}) which belongs to one of the classes C(i)-C(iii) s.t. $T_{m_n}(\rho_{m_n}) \to_d \mathcal{N}(0,1)$
- Conclusion: for any subsequence of $(T_n(\rho_n))$, there exists a further subsequence that converges to $\mathcal{N}(0,1)$ in distribution. Then $T_n(\rho_n) \to_d \mathcal{N}(0,1)$
- *Uniform size*: $(\rho_n)_{n\in\mathbb{N}}$ is an arbitrary approximating sequence of each element of the parameter space; Andrews, Cheng and Guggenberger (2020)

Instrument Selection

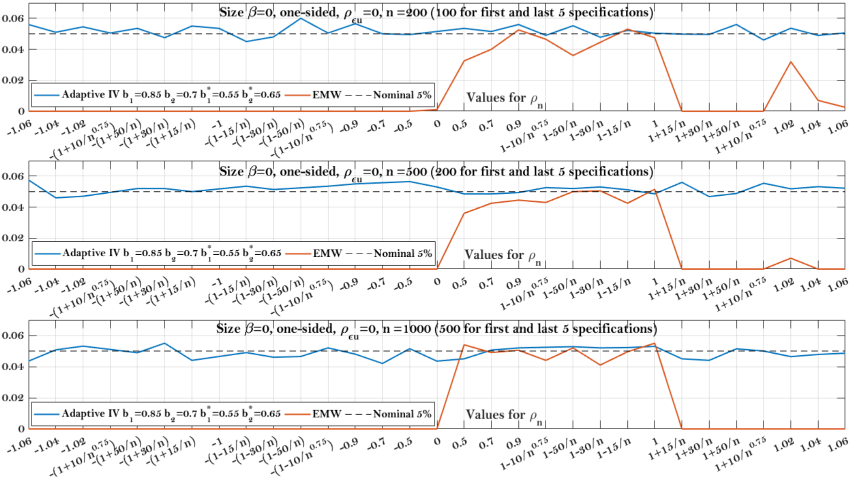
- We set $\varphi_{1n}^{-} = -\varphi_{1n}$ and $\varphi_{2n}^{-} = -\varphi_{2n}$
- Any values of φ_{1n} and φ_{2n} s.t. $\varphi_{1n} = 1 - 1/\kappa_{1n}$, $\varphi_{2n} = 1 + 1/\kappa_{2n}$ with $\kappa_{1n}, \kappa_{2n} \to \infty$, $\kappa_{1n}/n \to 0$, $\kappa_{2n}/n \to 0$: valid asymptotic inference
- But finite sample performance may vary considerably
- Choosing $\varphi_{1n} = 1 - 1/n^{b_1}, \varphi_{2n} = 1 + 1/n^{b_2}$: find values for b_1 and b_2 in $(0, 1)$
- We require instrument with good performance along ALL autoregressive regions without *a priori* knowledge
- A conservative approach: our procedure suffers worst finite sample distortion when $\rho_n = 1$ with large $\rho_{\varepsilon u}$
- We minimise the worst case scenario and select values that deliver satisfactory test size
- Grid of values for b_1 and b_2 with $|\rho_{\varepsilon u}| = 0.99$ $\Rightarrow b_1 = 0.85$ and $b_2 = 0.7$

Monte Carlo Simulations

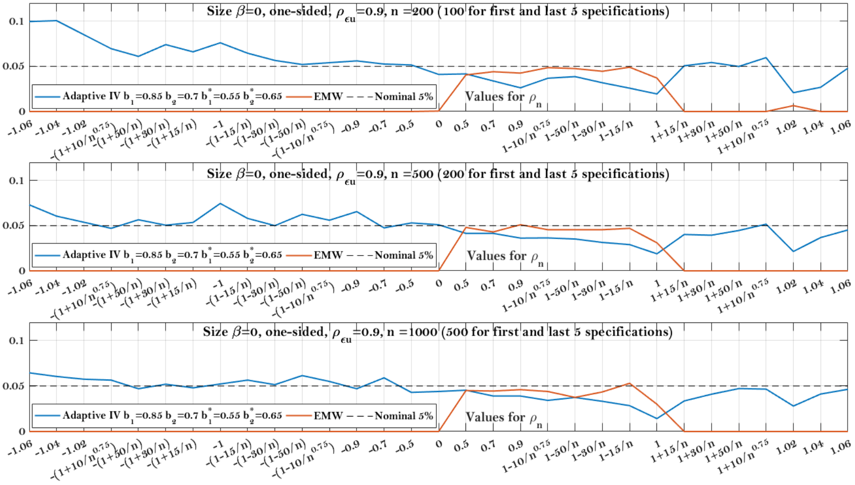
$$
\begin{aligned}
y_t &= \mu + \beta x_{t-1} + \varepsilon_t,\ x_t = \mu_x + \rho_n x_{t-1} + u_t,\ X_0 = 0,\ \mu_x = \mu_y = 0, \\
\varepsilon_t, u_t &\sim \mathcal{N}(0,1),\ \rho_{\varepsilon u} = \mathbb{E}(\varepsilon_t u_t),\ b_1 = 0.85,\ b_2 = 0.7, \\
\varphi_{1n} &= 1 - 1/n^{b_1}, \varphi_{2n} = 1 + 1/n^{b_2},\ \varphi_{1n}^- = -\varphi_{1n},\ \varphi_{2n}^- = -\varphi_{2n}, \\
\rho_{\varepsilon u} &\in \{-0.9, 0, 0.9\},\ n \in \{200, 500, 1000\} \\
\rho_n &\in \{\text{-1.06,-1.04,-1.02,-}(1+10/n^{0.75}),-(1+50/n),-(1+30/n), \\
&-(1+15/n),-1,-(1-15/n),-(1-30/n),-(1-50/n),-(1-10/n^{0.75}), \\
&-0.9,-0.7,-0.5,\ 0,\ 0.5,\ 0.7,\ 0.9,\ 1-10/n^{0.75},1-50/n,1-30/n,1-15/n, \\
&1,1+15/n,1+30/n,\ 1+50/n,\ 1+10/n^{0.75},\ 1.02,\ 1.04,\ 1.06\}
\end{aligned}
$$

- We compare empirical size and power for β against Elliott, Müller and Watson (2015) procedure
- We compare coverage and length of CIs for ρ_n against Andrews and Guggenberger (2014) procedure and for IRF ρ_n^h against Olea and Plagborg-Moller (2021)

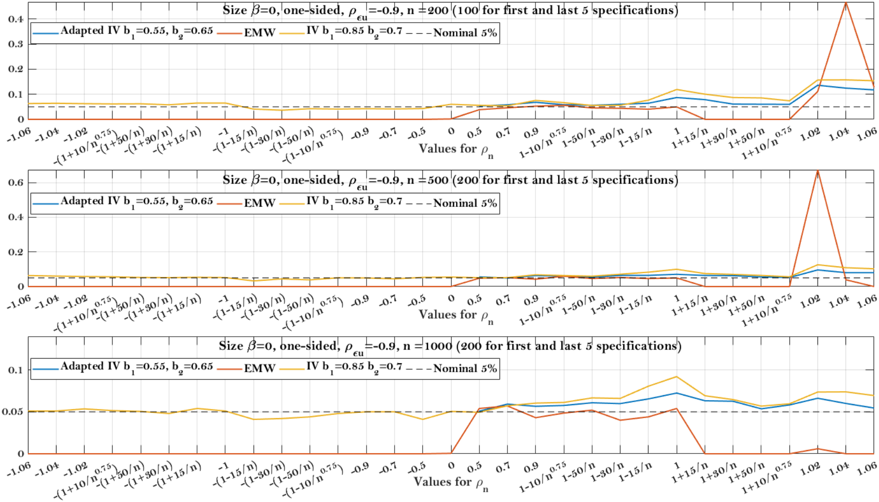
Monte Carlo Simulations: Size - zero correlation



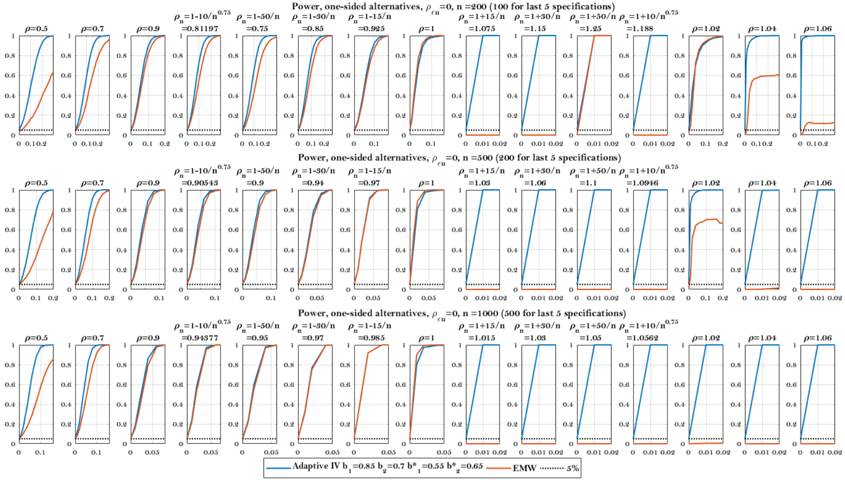
Monte Carlo Simulations: Size - positive correlation



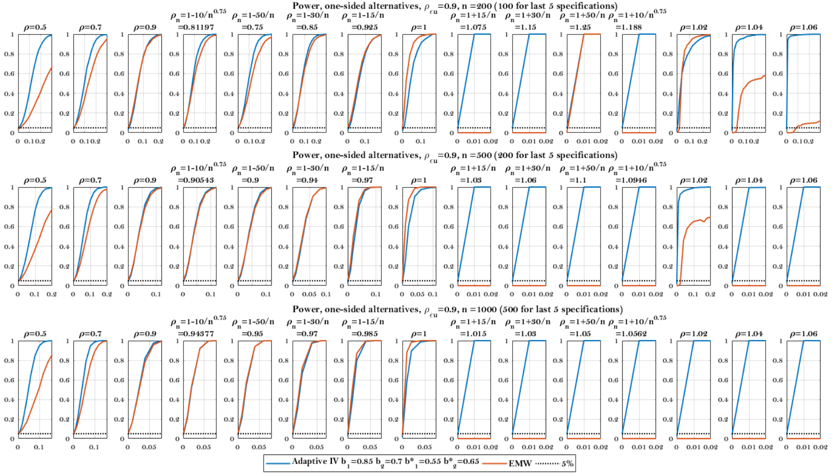
Monte Carlo Simulations: Size - negative correlation



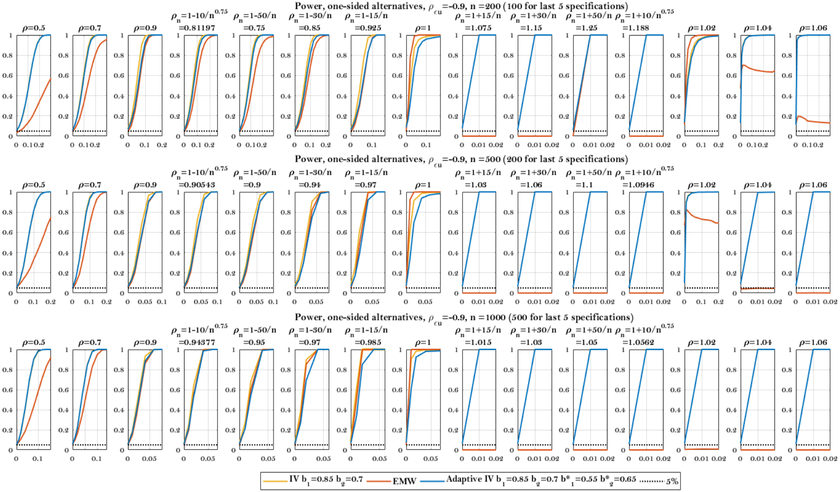
Monte Carlo Simulations: Power - zero correlation



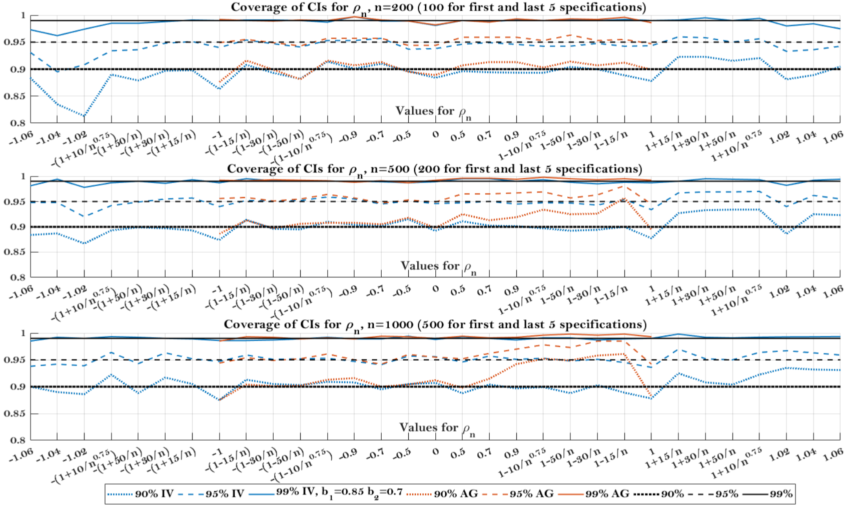
Monte Carlo Simulations: Power - positive correlation



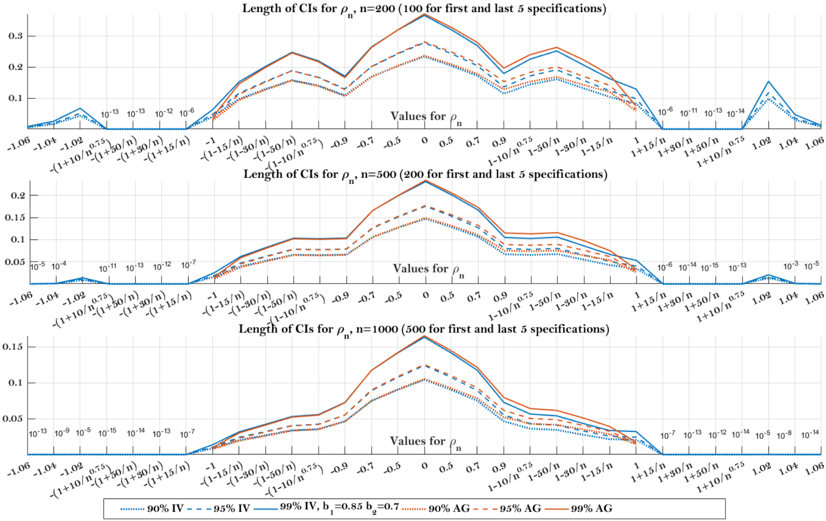
Monte Carlo Simulations: Power - negative correlation



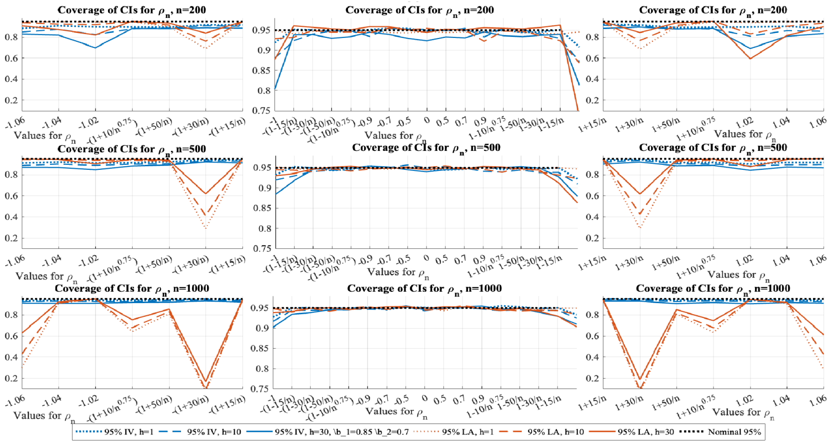
Coverage Rates



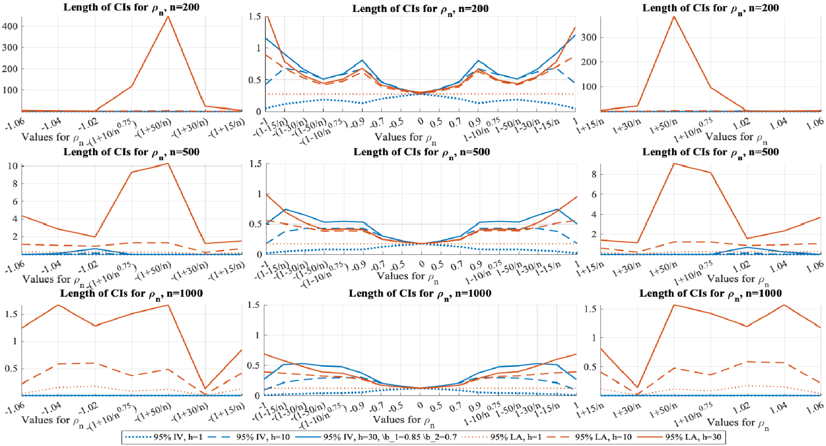
Length


Length of CIs for ρn, n=200 (100 for first and last 5 specifications)
Length of CIs for ρn, n=500 (200 for first and last 5 specifications)
Length of CIs for ρn, n=1000 (500 for first and last 5 specifications)
Values for ρn
90% IV
95% IV
99% IV, b1=0.85 b2=0.7
90% AG
95% AG
99% AG

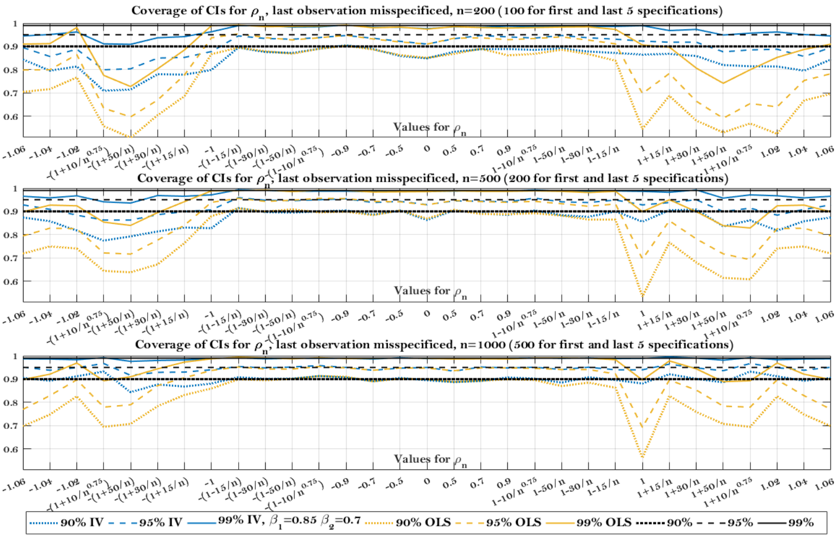
Coverage


Coverage of CIs for ρ_n, n=200
Coverage of CIs for ρ_n, n=200
Coverage of CIs for ρ_n, n=200
Coverage of CIs for ρ_n, n=500
Coverage of CIs for ρ_n, n=500
Coverage of CIs for ρ_n, n=500
Coverage of CIs for ρ_n, n=1000
Coverage of CIs for ρ_n, n=1000
Coverage of CIs for ρ_n, n=1000
0.2 0.4 0.6 0.8
0.75 0.8 0.85 0.9 0.95
-1.06 -1.04 -1.02 -(1+10/n$^{0.75}$) -(1+50/n) -(1+30/n) -(1+15/n)
-1 -(1-15/n) -(1-30/n) -(1-50/n) -(1-10/n$^{0.75}$) -0.9 -0.7 -0.5 0 0.5 0.7 0.9 1-10/n$^{0.75}$ 1-50/n 1-30/n 1-15/n
1+15/n 1+30/n 1+50/n 1+10/n$^{0.75}$ 1.02 1.04 1.06
Values for ρ_n
95% IV, h=1
95% IV, h=10
95% IV, h=30, \b_1=0.85 \b_2=0.7
95% LA, h=1
95% LA, h=10
95% LA, h=30
Nominal 95%

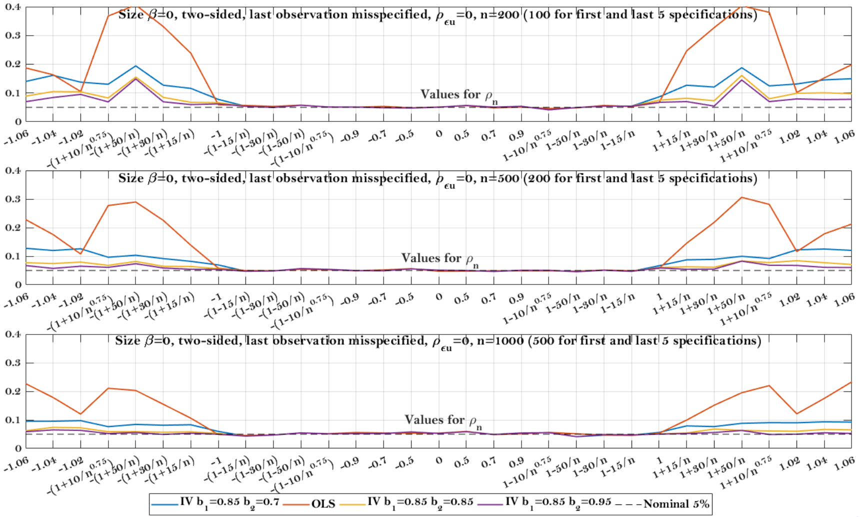
Length


Length of CIs for ρn, n=200
Length of CIs for ρn, n=500
Length of CIs for ρn, n=1000
Values for ρn
95% IV, h=1
95% IV, h=10
95% IV, h=30, \b_1=0.85 \b_2=0.7
95% LA, h=1
95% LA, h=10
95% LA, h=30

Failure of OLS in explosive region



Failure of OLS in explosive region



SIR model

- Kermack and McKendrick (1927)
- Various versions (SI, SIS and SIR models): widely used in the epidemiological literature
- continuous time nonlinear differential equation system
- discrete-time stochastic versions the model
- Basic ingredients:
    - total *infected*, *susceptible*, *recovered and deceased* individuals at time t, denoted by I_t, S_t, R_t, D_t
    - population of size N, $S_t + I_t + R_t + D_t = N$ for each $t \geq 0$
    - non-negative initial conditions S_0, I_0, R_0, D_0
    - no births or deaths by causes other than the epidemic
    - no heterogeneity: each individual equally likely to be infected
    - no the possibility of re-infection

Discretised SIR model

- The nonlinear discretised system:

$$\begin{aligned} I_{t+1} &= I_t\left(1+\frac{\theta}{N}S_t-\gamma-\delta\right) \\ S_{t+1} &= S_t\left(1-\frac{\theta}{N}I_t\right) \\ R_{t+1} &= R_t+\gamma I_t,\ \ D_{t+1}=D_t+\delta I_t \end{aligned}$$

- The model's parameters $\theta, \gamma, \delta \in (0, 1)$:
  - θ is the contraction rate (average individuals an infected person passes the infection to during Δt)
  - γ is the recovery and δ is the death rate
- Model dynamics driven by the *basic reproduction number* $r_0 = \theta/\left(\gamma+\delta\right)$
  - $r_0 \geq 1$: disease continues to spread
  - $r_0 < 1$ the infection rate will be contained

Linearisation and introduction of stochastic error

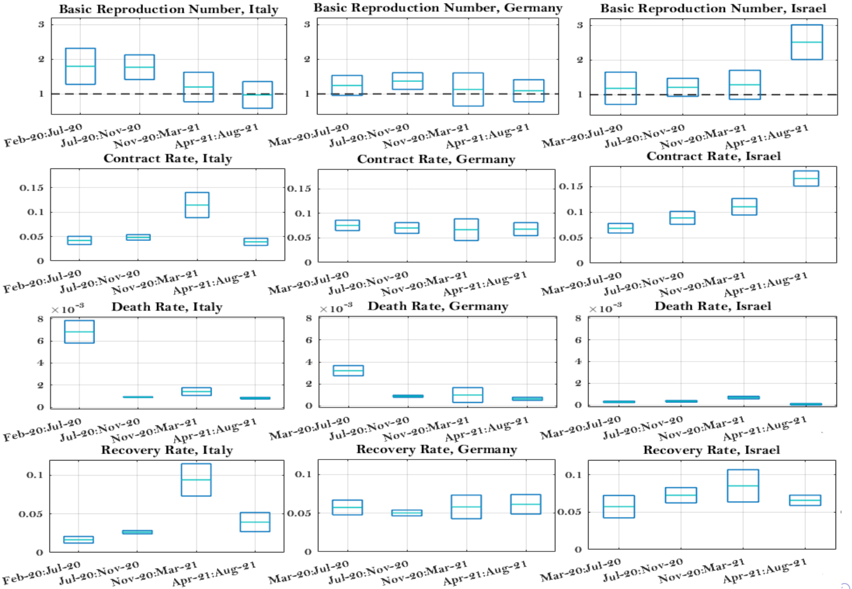
- Next generation matrix (NGM) approach: linearisation around DFE ($S = N, I = R = D = 0$); J Jacobian matrix at DFE
- Linearised system: $Y_t = JY_{t-1}$, with J lower triangular, $Y_t = [I_t, S_t, R_t, D_t]'$
- No births/deaths from other causes: $S_t = N - I_t - R_t - D_t$
- Introduction of a non-systematic error:

$$\begin{bmatrix} I_t \\ \Delta R_t \\ \Delta D_t \end{bmatrix} = \begin{bmatrix} 1+\theta-\gamma-\delta & 0 & 0 \\ \gamma & 0 & 0 \\ \delta & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{t-1} \\ R_{t-1} \\ D_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix}$$

- $r_0 = 1 + (\theta - \gamma - \delta) / \gamma \geq 1 \iff 1 + \theta - \gamma - \delta \geq 1$
- Dynamics of I_t: *AR(1) process* with $\rho = 1 + \theta - \gamma - \delta$
- Dynamics of R_t and D_t: *predictive regression* $\Delta R_t = \gamma I_{t-1} + u_{2t}$ and $\Delta D_t = \delta I_{t-1} + u_{3t}$
- OLS-based inference only valid for $\rho \in (0, 1)$

Empirical Application

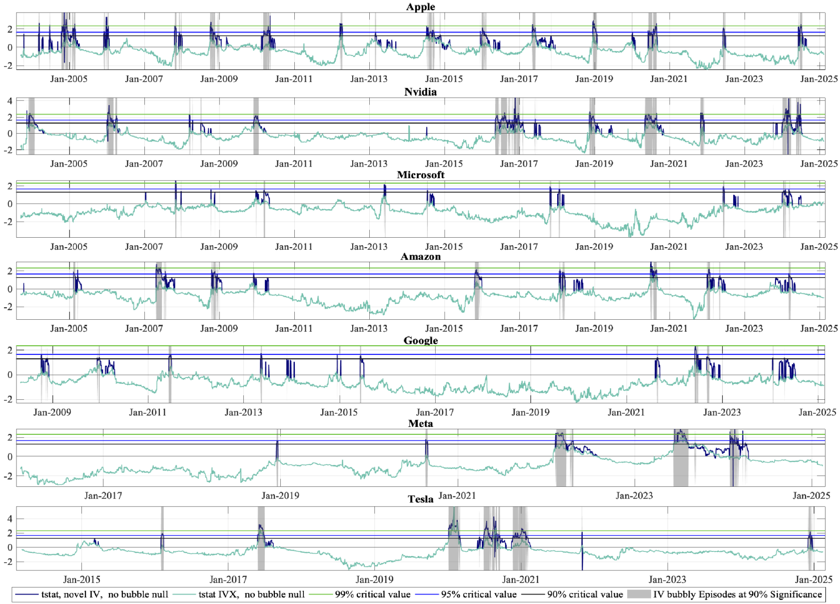
- We take the linear stochastic model to study the dynamics and transmission of covid-19
- We use data from the John Hopkins university database on Italy, Germany and Israel (more countries in the paper)
- We construct the series for removed as adding the series for recovered and dead from the disease
- For each country, we construct 4 subsamples


Basic Reproduction Number, Italy
Basic Reproduction Number, Germany
Basic Reproduction Number, Israel
Contract Rate, Italy
Contract Rate, Germany
Contract Rate, Israel
Death Rate, Italy
Death Rate, Germany
Death Rate, Israel
Recovery Rate, Italy
Recovery Rate, Germany
Recovery Rate, Israel
Feb-20:Jul-20
Mar-20:Jul-20
Jul-20:Nov-20
Nov-20:Mar-21
Apr-21:Aug-21
$\times 10^{-3}$

Bubbles in the Magnificent Seven's Stock Price

- Daily stock prices from Bloomberg: 04/01/2000-18/02/2025
- 12-month projected prices by analysts, IBES database
- Search for speculative bubbles in difference between realised and future prices
- One-sided unit root test, $H_0 : \rho = 1$, $H_1 : \rho > 1$
- Search with simple 300-day rolling window; more elaborate algorithms possible
- Our tstat is simple $\mathcal{N}(0,1)$ and easily amendable
- It is valid in purely explosive region, in contrast to alternatives which only allow for mildly explosive alternatives
- It remains valid even for stationary regions, which is useful for periods of bubble collapse and mean-reversion

Bubbles



Apple
Nvidia
Microsoft
Amazon
Google
Meta
Tesla
Jan-2005
Jan-2007
Jan-2009
Jan-2011
Jan-2013
Jan-2015
Jan-2017
Jan-2019
Jan-2021
Jan-2023
Jan-2025
tstat, novel IV, no bubble null
tstat IVX, no bubble null
99% critical value
95% critical value
90% critical value
IV bubbly Episodes at 90% Significance

Conclusion

- New IV procedure (based on novel instrument) for regressor with arbitrary persistence degree $\rho_n \to \rho \in (-\infty, \infty)$
- The resulting data-driven procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - $\mathcal{N}(0,1)$ inference (testing and CIs)
  - uniform size
  - first procedure for distribution-free inference in the purely explosive region
- Practical solution for applied researchers:
  - linear procedure, $\mathcal{N}(0,1)$ critical values
  - no need for pretesting, can handle all cases

Extensions

- Extension of uniform inference to richer dynamics and dimensions:
  - AR(p) and VAR(p) extensions;
  - truncated MA processes ((non)stationary long memory)
  - processes with changing persistence
- Idea: construct instrument to conform to standard limit theory even when original regressor does not
- Place a large class of processes with wide range of dynamics/ memory properties/ dimensions under common econometric framework
- Deliver standard inference uniformly over the parameter space
- Allow researchers to use $\mathcal{N}(0,1)$ and χ^2 critical values