$\kappa = 1$

Limit Theory and Tests for Infinite-Mean ACD

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Venice Time Series Workshop

What this talk is about (vs.1)

It's about sums of random variables:

$\sum_{i=1}^{D} x_i$ D deterministic and $\{x_i\}$ sequence of r.v.'s

$\sum_{i=1}^{R} x_i$ R random and depends on $\{x_i\}$

What this talk is about (vs.2)

It's about a corollary (Engle and Russell, 1998*ecma*):

COROLLARY TO LEE AND HANSEN (1994): *If*:
(A) $E_{i-1}(x_i) \equiv \psi_{0,i} = \omega_0 + \alpha_0 x_{i-1} + \beta_0 \psi_{0,i-1}$;
(B) $\varepsilon_i \equiv x_i / \psi_{0,i}$ *is* (i) *strictly stationary and ergodic*, (ii) *nondegenerate*, (iii) *has bounded conditional second moments*, (iv) $\sup_i E[\ln(\beta_0 + \alpha_0 \varepsilon_i) \mid F_{i-1}] < 0$ *a.s.*;
(C) $\theta_0 \equiv (\omega_0, \alpha_0, \beta_0)$ *is in the interior of* Θ;

(D) $$L(\theta) = -\sum_{i=1}^{N(T)} \left\{ \log(\psi_i) + \frac{x_i}{\psi_i} \right\} \quad \textit{where}$$

$$\psi_i = \omega + \alpha x_i + \beta \psi_{i-1} \quad \textit{for } i > 1,$$
$$\psi_i = \omega/(1-\beta) \quad \textit{for } i = 1;$$

then:
(a) *the maximizer of L will be consistent and asymptotically normal with a covariance matrix given by the familiar robust standard errors from Lee-Hansen; and*
(b) *the model can be estimated with ARCH software by taking* $\sqrt{x_i}$ *as the dependent variable and setting the mean to zero.*

What this talk is about (vs.3)

A time interval (for example, a trading day or a year):

$$\text{period: } [0, t] \qquad \text{events: } 0 = t_0 < t_1 < t_2 < \ldots < t_{n(t)} \leq t$$

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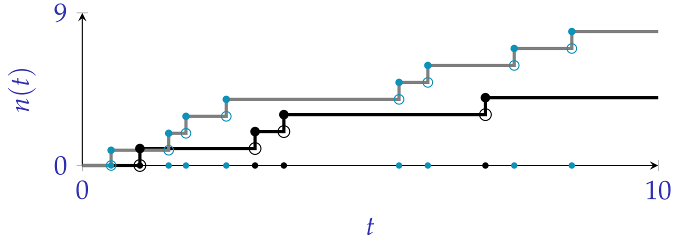

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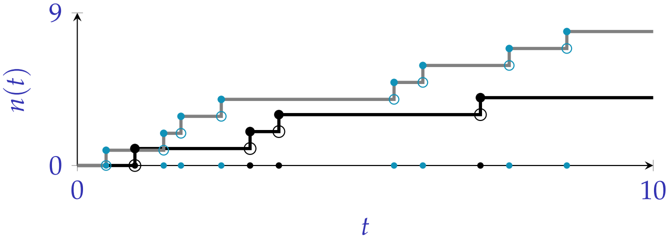

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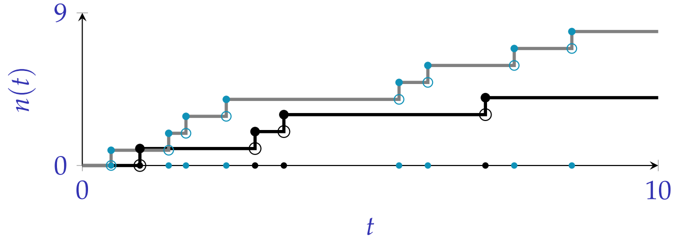

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Focus: ACD model (Engle & Russell, 1998[ecma]; Engle, 2000[ecma])

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the simplest ACD model for x_i is given by

$$\begin{aligned} x_i &= \psi_i \varepsilon_i, \qquad i = 1, 2, \ldots, n(t), \\ \psi_i &= \omega + \alpha x_{i-1} \ (+\beta \psi_{i-1}) \end{aligned}$$

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Integrated ACD [IACD] '$\kappa = 1$': $\alpha = 1$ ($\alpha + \beta = 1$)

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This talk: inference and theory for $\kappa = 1$.

ACD properties

Consider the simple ACD with ε_i i.i.d. exponentially distributed:

$$x_i = \psi_i \varepsilon_i, \qquad \psi_i = \omega + \alpha x_{i-1}$$

The coefficient α determines properties of x_i.

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α	Stationary and Ergodic	Moments	Tail index
$0 \leq \alpha < 0.7$	$\surd$	$\mathbb{E}[x_i^2] < \infty$	$\kappa > 2$
$0.7 \leq \alpha < 1$	$\surd$	$\mathbb{E}[x_i] < \infty$	$1 < \kappa \leq 2$
$\boldsymbol{\alpha = 1}$	$\surd$	$\mathbb{E}[x_i] = \infty$	$\boldsymbol{\kappa = 1}$
$1 < \alpha < e^{\gamma} \simeq 1.8$	$\surd$	$\mathbb{E}[x_i] = \infty$	$\kappa < 1$

Recall tail index κ: $\mathbb{P}(x_i > x) \sim x^{-\kappa}$.

Recall ARCH(1) stationary: $\alpha < 3.6$.

ACD stochastic properties

Consider the 'GARCH' ACD with ε_i i.i.d. exponentially distributed:

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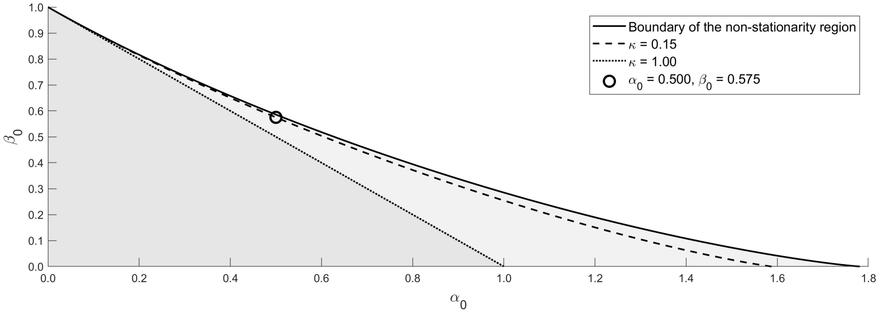
	$\alpha_0 + \beta_0$	Stationary and Ergodic	Moments	Tail index
nice	$\alpha_0 + \beta_0 < 1$	$\surd$	$\mathbb{E}[x_i] < \infty$	$\kappa > 1$
integrated	$\alpha_0 + \beta_0 = 1$	$\surd$	$\mathbb{E}[x_i] = \infty$	$\kappa = 1$
wild	$\alpha_0 + \beta_0 > 1$	$\surd$	$\mathbb{E}[x_i] = \infty$	$\kappa < 1$

ACD Tail index

Stationarity and ergodicity | Tail index:

$$\mathbb{E}[\log(\alpha_0\varepsilon_i + \beta_0)] < 0 \qquad \Big| \qquad h(\kappa) = \mathbb{E}[(\alpha_0\varepsilon_i + \beta_0)^\kappa] = 1$$



Time span and number of events

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As the length of the period grows, so does the number of events:

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And for the integrated case? **New result**: $n(t)$ grows slower than t:

$$\alpha_0 + \beta_0 = 1 \; (\kappa = 1): \qquad \frac{n(t)}{t/\log(t)} \xrightarrow{p} \text{constant}$$

ACD stochastic properties | Repeated

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In short:

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integrated	$\alpha_0 + \beta_0 = 1$	$\mathbb{E}[x_i] = \infty$	"small"	$\kappa = 1$
wild	$\alpha_0 + \beta_0 > 1$	$\mathbb{E}[x_i] = \infty$	"smaller"	$\kappa < 1$

ACD QMLE

The exponential log-likelihood function is given by

$$L(\omega,\alpha,\beta) = -\sum_{i=1}^{n(t)} \left(\log\psi_i + x_i/\psi_i\right), \qquad \psi_i = \omega + \alpha x_{i-1} + \beta\psi_{i-1}$$

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- ARCH(1) theory: asymptotic normality if $\{x_i\}$ is strictly stationary.

Main results

Consider behavior of $\hat{\theta} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})'$ as $t \to \infty$ (and hence $n(t) \to \infty$ a.s.):

$\alpha_0 + \beta_0 < 1$	$\kappa > 1$	$\sqrt{t}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mu\sigma_\varepsilon^2\Omega^{-1})$	$\mu = \mathbb{E}[x_i]$

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$\alpha_0 + \beta_0 = 1$	$\kappa = 1$	$\sqrt{\frac{t}{\log t}}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, c\sigma_\varepsilon^2 \Omega^{-1})$	c constant

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  - Gaussian with rate $\sqrt{t}$ if $\kappa > 1$ ($\alpha_0 + \beta_0 < 1$)
  - Mixed Gaussian with rate $\sqrt{t^\kappa}$ if $\kappa < 1$ ($\alpha_0 + \beta_0 > 1$).
  - **NEW:** Gaussian but with rate $\sqrt{t/\log t}$ if $\kappa = 1$ ($\alpha_0 + \beta_0 = 1$).

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  - **NEW:** Gaussian but with rate $\sqrt{t/\log t}$ if $\kappa = 1$ ($\alpha_0 + \beta_0 = 1$).
- Different from (G)ARCH: $\sqrt{t}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}$ for any κ.
- Engle and Russell (1998[ecma]): $\hat{\theta}$ asymptotically normal irrespective of κ.

For the ACD given by

$$x_i = \psi_i \varepsilon_i, \qquad \psi_i = \omega + \alpha x_{i-1} + \beta \psi_{i-1}, \qquad i = 1, 2, 3, \ldots, n(t)$$

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Theorem

For $\alpha_0 + \beta_0 = 1$,

$$\begin{aligned} t\text{-}statistic &= \frac{\hat{\alpha} + \hat{\beta} - 1}{st.err.} \xrightarrow{d} \mathcal{N}(0,1), \\ \text{LR} &\xrightarrow{d} \chi_1^2. \end{aligned}$$

Note: QLR $\xrightarrow{d} \mathbb{V}(\varepsilon_i) \times \chi_1^2$.

Essential Insights: likelihood analysis | I/III

Recall:

nice	$\alpha_0 + \beta_0 < 1$	$\kappa > 1$	$\frac{n(t)}{t} \xrightarrow{p}$ constant
integrated	$\alpha_0 + \beta_0 = 1$	$\kappa = 1$	$\frac{n(t)}{t/\log(t)} \xrightarrow{p}$ constant
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In all asymptotic arguments, we need convergence of the information, $\mathcal{I} = -\partial^2 L(\theta)/\partial\theta\partial\theta'$ where $\theta = (\omega, \alpha, \beta)'$:

$$\mathcal{I} = \sum_{i=1}^{n(t)} \xi_i, \qquad \xi_i = g(\varepsilon_i, \varepsilon_{i-1}, \dots)$$

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corresponding to the nice, integrated and wild cases above. Crucial: different rates and random limit of information.

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$$\frac{1}{\sqrt{t}}\mathcal{S} \xrightarrow{d} \mathcal{N}, \qquad \frac{1}{\sqrt{t/\log t}}\mathcal{S} \xrightarrow{d} \mathcal{N}, \qquad \frac{1}{\sqrt{t^\kappa}}\mathcal{S} \xrightarrow{d} \mathcal{MN}$$

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Note the $\mathcal{MN}$

Essential Insights: likelihood analysis | III/III

With $\mathcal{S} = \partial L(\theta)/\partial\theta$ and $\mathcal{I} = -\partial^2 L(\theta)/\partial\theta\partial\theta'$:

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Theorem

If $\alpha_0 + \beta_0 < 1$, *then* $\frac{n(t)}{t} \xrightarrow{p}$ *constant and*

$$(\mathcal{S}/\sqrt{t}, \mathcal{I}/t) \xrightarrow{d} (\mathcal{N}, constant)$$

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If $\alpha_0 + \beta_0 > 1$, *then* $\frac{n(t)}{t^\kappa} \xrightarrow{d}$ *random variable and (jointly; cf. Sweeting, 1992[aos])*

$$(\mathcal{S}/\sqrt{t^\kappa}, \mathcal{I}/t^\kappa) \xrightarrow{d} (\mathcal{MN}, random\ information)$$

Essential Insights: $n(t)$ | I/V

Recall

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\begin{aligned}
n(t) &= \max\{k : x_1 + \cdots + x_k \leq t\} = \max\{k : s_k \leq t\} \to \infty \text{ (a.s.) as } t \to \infty, \\
s_n &= \sum_{i=1}^{n} x_i
\end{aligned}
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and $s_{n(t)} = \sum_{i=1}^{n(t)} x_i \simeq t$.

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- The behavior of $n(t)$, along with $\sum_{i=1}^{\lfloor n\cdot \rfloor} \xi_i$, is the main ingredient.

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- $n(t) \to \infty$ (a.s.) is **not sufficient** for CLTs to apply to sums like $\sum_{i=1}^{n(t)} \xi_i$ (if they apply at all).
- The behavior of $n(t)$, along with $\sum_{i=1}^{\lfloor n\cdot \rfloor} \xi_i$, is the main ingredient.
- The key equality to establish the behavior of $n(t)$ is the 'inversion' formula

$$\mathbb{P}(n(t) \ge k) = \mathbb{P}(s_k \le t)$$

relating the renewal process $n(t)$ to the random walk s_n.

Essential Insights: $n(t)$ | II/V

- Intuition: if x_i has finite mean $\mu = \mathbb{E}[x_i]$ (and is geometrically ergodic), i.e. $\kappa > 1$:

$$\frac{n(t)}{t} = \left(\frac{t}{n(t)}\right)^{-1} \approx \left(\frac{\sum_{i=1}^{n(t)} x_i}{n(t)}\right)^{-1} \xrightarrow{p} \mu^{-1}.$$

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$$\mathbb{P}\left(\frac{n(t)}{t} \geq z\right) = \mathbb{P}(n(t) \geq zt) \approx \mathbb{P}(s_{\lfloor zt \rfloor} \leq t) \approx \mathbb{P}\left(\frac{s_{\lfloor zt \rfloor}}{\lfloor zt \rfloor} \leq \frac{1}{z}\right) \to \mathbb{I}(z \leq \mu^{-1})$$

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- If x_i has infinite mean, i.e. $\kappa \leq 1$, this result breaks down:

$$\frac{n(t)}{t} \approx \left(\frac{\sum_{i=1}^{n(t)} x_i}{n(t)}\right)^{-1} = \left(\frac{s_{n(t)}}{n(t)}\right)^{-1} \overset{a.s.}{\to} 0$$

as $n(t)$ does not match the variation in the sum of the x_i's.

Essential Insights: $n(t)$ | III/V

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- This follows since

$$\begin{aligned} \mathbb{P}\left(\frac{n(t)}{t^\kappa} \geq z\right) &= \mathbb{P}(n(t) \geq zt^\kappa) \approx \mathbb{P}(s_{\lfloor zt^\kappa \rfloor} \leq t) \\ &= \mathbb{P}\left(\frac{s_{\lfloor zt^\kappa \rfloor}}{\lfloor zt^\kappa \rfloor^{1/\kappa}} \leq \frac{t}{\lfloor zt^\kappa \rfloor^{1/\kappa}}\right) \approx \mathbb{P}\left(\frac{s_{\lfloor zt^\kappa \rfloor}}{\lfloor zt^\kappa \rfloor^{1/\kappa}} \leq \frac{1}{z^{1/\kappa}}\right) \\ &\to \mathbb{P}(\eta_\kappa \leq z^{-1/\kappa}) = \mathbb{P}(\eta_\kappa^{-\kappa} \geq z) \end{aligned}$$

Essential Insights: $n(t)$ | IV/V

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- In the i.i.d. case, sums typically require both scaling and centering:

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with d_n of order $n \log n$.

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- Because $\kappa = 1$, $\mathbb{P}(x_i > x) \sim c_0 x^{-1}$, and we expect

$$a_n = n, \qquad d_n = c_0 n \log n.$$

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  However, we could not prove (1).
- What we could prove is the law of large numbers under a stronger normalization

$$\frac{s_n - c_0 n \log n}{n \log n} \xrightarrow{p} 0,$$

  which implies

$$\frac{s_n}{n \log n} \xrightarrow{p} c_0.$$

Essential Insights: $n(t)$ | V/V

- However, setting $m_t = \left\lfloor z \frac{t}{\log t} \right\rfloor$,

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&= \mathbb{P}\left(\frac{s_{m_t} - c_0 m_t \log m_t}{m_t \log m_t} + c_0 \leq \frac{t}{m_t \log m_t}\right) \\
&\approx \mathbb{P}\left(\frac{s_{m_t} - c_0 m_t \log m_t}{m_t \log m_t} + c_0 \leq \frac{1}{z}\right) \\
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which yields the new renewal result

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- To the best of our knowledge, this conclusion is new even in the i.i.d. case.

Empirics

The most applied ACD model is the ACD/GARCH(1,1):

$$x_i = \psi_i \varepsilon_i, \qquad \psi_i = \omega + \alpha x_{i-1} + \beta \psi_{i-1}$$

Tail index empirically:

κ	Data
> 2	IBM: Engle and Russell (1998)
$\simeq 1$	Crypto ETFs: this talk
< 2	DJ: Embrechts, Liniger and Lin (2011)
< 1	SPY: Cavaliere, Mikosch, Rahbek and Vilandt (2024)

Data example: ETFs tracking cryptocurrency prices

Exchange-traded funds (ETFs) tracking cryptocurrency prices:
Grayscale Bitcoin Mini Trust (**BTC**), Grayscale Ethereum Mini Trust (**ETH**), Grayscale Bitcoin Trust (**GBTC**), Grayscale Ethereum Trust (**ETHE**) and Bitwise Bitcoin (**BITB**).

Time period $[0, t]$:
Jan–Feb, 2025: NASDAQ (9:30AM to 4PM), trading times in seconds.
35 trading days with $6.5 \cdot 60^2 = 23{,}400$ seconds of trading every day, hence $t = 35 \cdot 23{,}400 = 819{,}000$ seconds.

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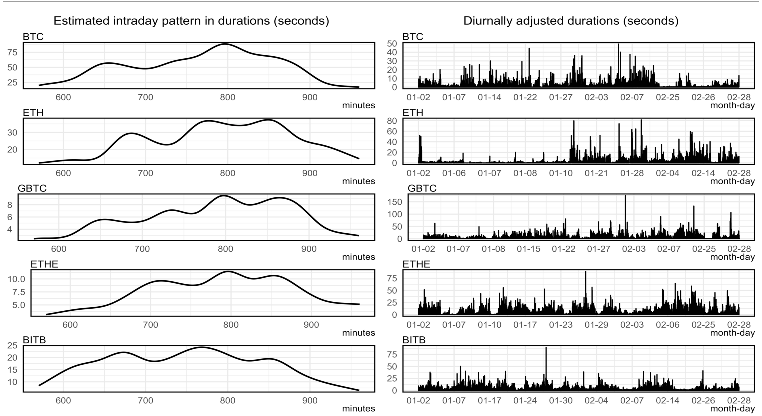

Number of trades $n(t)$:
BTC 19,366; ETH 35,492; GBTC 157,620; ETHE 120,104; BITB 51,917.

Data example:

For the ACD(1,1)

$$x_i = \psi_i \varepsilon_i, \qquad \psi_i = \omega + \alpha x_{i-1} + \beta \psi_{i-1}, \qquad i = 1, 2, 3, \ldots, n(t)$$

We find for the cryptocurrency trading data:

$$\mathbb{E}[x_i] = \infty, \qquad \text{or } \kappa \leq 1.$$

Conclusion

- **New complete estimation and inference theory for ACD models**, including testing for integrated ACD.
- **Tail index κ is important** for both asymptotic and finite-sample properties:

$$\sqrt{t}, \qquad \sqrt{t/\log t}, \qquad \text{and} \qquad \sqrt{t^{\kappa}} \text{ rates.}$$

- **Testing integrated ACD:**

$$\text{LR and } t\text{-ratio} \to \chi^2 \text{ and } \mathcal{N}(0,1) \text{ respectively.}$$

- Ongoing work: **bootstrap** inference; **uniformity** issues.
- **References:**

Cavaliere, Mikosch, Rahbek and Vilandt, 2024 [joe]
Cavaliere, Mikosch, Rahbek and Vilandt, 2025 [ecma]
Cavaliere, Mikosch, Rahbek and Vilandt, 2026 [jrssb]